COMPLEX CONICS AND THEIR REAL REPRESENTATION

BY

BENJAMIN ERNEST MITCHELL

DISSERTATION

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY, IN THE FACULTY OF PURE SCIENCE, COLUMBIA UNIVERSITY

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INTRODUCTION.

1. *Historical.*—The introduction of the imaginary quantity, or the complex quantity comprehending both the real and the imaginary, into analysis had the effect not only of extension and generalization but also in many cases of simplification. Such results in the realm of geometry have not yet been fully realized. In the preface to his "Einführung in die analytische Geometrie," Kowalewski says: "Eine grosse Schwierigkeit in der analytischen Geometrie ist die exakte Behandlung des Imaginären."

But the incorporation of the imaginary in geometry does not require any more of reconstruction and readjustment than it did in the case of analysis.[1] "A satisfactory theory of imaginary quantities of the ordinary algebra . . . with difficulty obtained recognition in the first third of this century . . . , it . . . was not sought for or invented—it forced itself, unbidden, upon the attention of mathematicians, and with its rules already formed."[2]

No sooner had the imaginary won its rightful place in analysis at the hand of Gauss and Cauchy than it began to knock at the door of geometry. Indeed before its full recognition in analysis there had appeared the geometric method of representing the imaginary quantity due to Argand and Wessel. But the imaginary in geometry must play the rôle of element of structure comparable to that of number in analysis.[3] The history of its development is of intense interest.

[1] Convention. Toute expression ayant un sens géométrique quand les éléments dont elle depend sont réels conservera, par definition, le même nom quand quelques-uns de ces éléments deviendront imaginaries. Niewenglowski, Cours de Géométrie Analytique, p. 114, Old Edition.

[2] Gibbs: "On Multiple Algebra," Proc. Am. Asso. Adv. of Sci., 1886.

[3] Cf. C. A. Scott: "On Von Staudt's Geometrie der Lage," Math. Gazette, Vol. 1, p. 307.

Out of the school of Monge came a class of mathematicians, beginning with Poncelet and culminating in Von Staudt, who were somewhat exclusive in their methods. On the principle, geometry for geometers and geometry all-sufficient and self-sufficient, they set about to build up a body of doctrine wholly independent of analysis. The principals in this program were Poncelet, Chasles, Steiner and Von Staudt. The imaginary enters through the so-called Principle of Continuity and makes its first appearance in Chasles' "Traité de Géométrie Supérieure" (1852). Regarding the second of the three advantages which he claims for his geometry he says: "Je veux parler de la généralité dont sont empreints tous les résultats de la géométrie analytique, où l'on ne fait acception ni *des différences de positions relatives* des diverses parties d'une figure, ni des circonstances de réalité ou *d'imaginarité* des parties, qui, dans la construction générale de la figure, peuvent être indifféremment réelles ou imaginaries. Ce caractère spécifique de l'Analyse se trouve dans notre Géométrie."

But the investigations of Poncelet and Chasles had their origins in analysis and in their completed forms were not free from analytical considerations. To George Karl Christain Von Staudt belongs the honor of constructing independently of analysis a geometry involving imaginary elements. Thus according to Von Staudt: Two conjugate imaginary points may always be considered as the double points of an (elliptic) involution on a real line; and as (in analysis) we pass from an imaginary number to its conjugate by changing i to $-i$, so (in geometry) we may distinguish the two imaginary points by associating them respectively with the two senses of the line.

Now the most essential or characteristic ideal of geometry is to render all configurations visualizable, intuitive: the desideratum is, to use the German, "Anschaulichkeit." Whilst Von Staudt's purely projective methods were theoretically sufficient, yet they were found to be in use cumbrous and complicate; accordingly we find the diverging lines of analysis and geometry beginning to change direction and to come together. "From this moment a brilliant period opens for geometrical research of every kind. Analysts interpret all their results and set to work to translate them by constructions. Geometers endeavor to discover in every question some general principle—in most cases impossible to prove without the aid of analysis."[1] This was but reflecting the spirit of the great Monge who "has shown from the outset . . . that the alliance between analysis and geometry was useful and fruitful and that perhaps their alliance was a condition of the success for both of these branches of mathematics."[2] For example, in the conclusion of his lecture "On the Real Shape of Algebraic Curves and Surfaces," as interpreted by geometric models and Riemann surfaces, Klein says: "These methods give us the actual mental image of the configuration under consideration, and this I consider the most essential in all true geometry."[3]

This ideal of geometry is entirely consistent with the ideal of mathematics as presented by Von Staudt himself: "Indem die Mathematik darnach strebt, Ausnahmen Von Regeln zu beseitigen und verschiedene Sätze aus einem Gesichtspunkte aufzufassen, wird sie häufig genöthigt, Begriffe zu erweitern oder neue Begriffe aufzustellen, was beinahe immer einen Fortschritt in der Wissenschaft bezeichnet."[4]

In these ideals we have the spirit and aim of the great program proposed by Professor Study in his lectures and elsewhere,[5] a program in accordance with which on the one hand we are not to be hampered by assumptions regarding reality or non-reality and on the other hand all configurations whether real or imaginary are to receive intuitive representation.

In his work "Vorlesungen über ausgewählte Gegenstände der Geometrie, erstes Heft: Ebene Analytische Kurven und zu ihnen gehörige Abbildungen" Study has blazed a path through the great domain contemplated by his program. My aim in this

[1] Darboux: "A Study of the Development of Geometric Methods," Congress of Arts and Science, St. Louis, 1904.

[2] Darboux, supra.

[3] The Evanston Colloquium. Lectures on Mathematics, Lecture IV.

[4] "Beiträge zur Geometrie der Lage," Vorwort.

[5] "Vorlesungen über ausgewählte Gegenstände der Geometrie" (1911), "Zur Differential-geometrie der analytischen Curven," and "Die natürlichen Gleichungen der analytischen Curven im Euclidischen Raume," Trans. Am. Math. Soc., vols. 10, 11.

paper, as indicated by its title, is to apply his general method to an interesting and important detail.

2. *The Laguerre-Study Representation of the Imaginary.*—There is a great variety of ways of representing the imaginary element by a real figure.[1] The most efficient for purposes of analysis is that due to Laguerre[2] extended and developed by Study.[3]

The two families of minimal lines, right- and left-sided, to use Study's term, have for equations

$$\xi + i\eta = \text{const.} \quad \text{and} \quad \xi - i\eta = \text{const.}$$

Where ξ and η are the rectangular cartesian coordinates of the ∞^4 finite complex points in a projective plane. These lines through a point (ξ', η') of the plane have for equations:

$$\xi + i\eta = \xi' + i\eta' \quad \text{and} \quad \xi - i\eta = \xi' - i\eta'.$$

On each of these lines there is one and only one real point. Taking (x', y') for the coordinates of the real point on the second and (u', v') those for the real point on the first, we have:

$$\xi' + i\eta' = u' + iv' \quad \text{and} \quad \xi' - i\eta' = x' - iy'.$$

Considering $u' + iv'$ and $x' + iy'$ as geometric pictures of two gaussian numbers w' and z' we write

$$\text{(I)} \qquad \xi' + i\eta' = w' \quad \text{and} \quad \xi' - i\eta' = \bar{z}',$$

where $\bar{z}' = x' - iy'$, the conjugate of z'. It is agreed to take the two real points z' and w' of the two gaussian planes as the Real Representation of the complex point (ξ', η') of the cartesian plane. Study symbolizes this representation by $z \rightarrow w$, and calls it the First Picture (das erste Bild) of the imaginary point.

Again we have for the conjugate point $(\bar{\xi}', \bar{\eta}')$ of (ξ', η')

$$\left.\begin{aligned} \xi + i\eta &= \bar{\xi}' + i\bar{\eta}', \\ \xi - i\eta &= \bar{\xi}' - i\bar{\eta}'; \end{aligned}\right\} \text{ consequently } \left\{\begin{aligned} \bar{\xi}' + i\bar{\eta}' &= z', \\ \bar{\xi}' - i\bar{\eta}' &= \bar{w}'. \end{aligned}\right.$$

Thus the picture of $(\bar{\xi}', \bar{\eta}')$ is $w' \rightarrow z'$.

[1] Encyklopädie der Mathematischen Wissenschaften, III, *AB*, 4*a*, 13–16.
[2] Oeuvres, Tome II, pp. 89–98.
[3] "Vorlesungen," p. 9.

Hence by virtue of relation (I) a perfect correspondence is set up between the totality of finite complex points of the plane and the totality of finite real point-pairs: to any complex point corresponds uniquely a real point-pair, and vice versa.[1]

The cartesian plane (ξ, η) and the picture planes (z) and (w) may be considered superposed or not; in either case they are to be considered distinct. The ∞^2 real points of the plane have $w = z$; that is, if the planes are considered coincident, these points are their own pictures.

[1] "Vorlesungen," p. 10.

CHAPTER I.

REAL CONICS.

3. *Some Simple Cases.*—If we have given an equation,

$$\varphi(\xi, \eta) = 0,$$

ξ, η being rectangular cartesian coordinates, there is simultaneously given, by virtue of the relations

$$\xi + i\eta = w, \qquad \xi - i\eta = \bar{z}$$

$$\text{(1)} \qquad \text{or} \qquad \xi = \frac{w + \bar{z}}{2}, \qquad \eta = \frac{w - \bar{z}}{2i},$$

a corresponding equation

$$f(w, \bar{z}) = 0,$$

setting up a correspondence between the two picture planes. This latter equation, being a relation between two complex variables, expresses a conformal transformation between the two planes. Since one of the variables appears in its conjugate form, angles under the transformation, while preserved in magnitude, are reversed in sense. Such a transformation Study calls an improper (uneigentlich) conformal transformation. We shall refer to such a transformation as *reverse conformal.*

If the coefficients of φ are real the curve is called real, although it may contain no real points. Compare § 4 of this chapter, and Study, p. 46.

Accordingly, for the sake of completeness and reference, we state some simple cases without giving the proof, and develop others.

(*A*) *The picture* of the ∞^2 points on a real line, $a\xi + b\eta + c = 0$, a, b, c being real,[1] is the ensemble of point pairs symmetric (the

[1] We shall observe the following notation throughout this paper: Complex quantities shall be represented by Greek characters, α, λ, ξ, etc., and real quantities by Latin characters a, l, x, etc., with the single exception w and z; these being complex and equivalent to $u + iv$ and $x + iy$ respectively.

one the reflection of the other) with respect to the real branch of the line. (Study, p. 25.)

(*B*) *The picture* of the ∞^2 points on a real circle is the ensemble of point-pairs symmetric (geometrically inverse) with respect to the circle. (Study, p. 36.)

Study gives the real ellipse as an example in his lectures on this subject. I shall, however, consider this case along with the real hyperbola and real parabola, the method used differing from Study's in no essential respect.

4. *The Real Ellipse.*—We have two cases, (1) where the ellipse has a real branch or arc, (2) where the ellipse has no real branch or arc.[1]

The equations corresponding to these two cases are of course

$$(1)\qquad \frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = \pm 1, \qquad a > b.$$

If we write $\epsilon = 1$ or i then $\epsilon^2 = \pm 1$ and we have

$$(2)\qquad \frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = \epsilon^2.$$

Form the pencil of lines

$$(3)\qquad \eta = \frac{b}{a}\tau(\xi + \epsilon a), \qquad \tau = s + it.$$

This line has one fixed intersection with the ellipse, $(-\epsilon a, 0)$, and one free intersection. The range of τ through the domain of real numbers, s arbitrary, $t = 0$, gives the ∞^1 real points on the ellipse $\epsilon = 1$. The same range of τ gives ∞^1 complex points of the ellipse $\epsilon = i$. The complete variation of τ, s and t assuming all possible real values, gives the ∞^2 points of the conic.

Expressing ξ and η in terms of the parameter τ, using equations (2) and (3),

$$(4)\qquad \xi = \epsilon a\frac{1-\tau^2}{1+\tau^2}, \qquad \xi + \epsilon a = \frac{2\epsilon a}{1+\tau^2}; \qquad \eta = \frac{2\epsilon b\tau}{1+\tau^2}.$$

[1] In investigations in this field, where a *curve* is considered as consisting of ∞^2 points, a configuration consisting of ∞^1 points is called by Segre *filo*, which Study translates, *Faden*, that is *thread*. The difference, then, in these two ellipses is that one has a real *thread* and the other has not.

Then for w and z in terms of the parameter we have:

$$(4') \quad \begin{aligned} w &= \xi + i\eta = -\epsilon a + 2\epsilon \frac{a + ib\tau}{1 + \tau^2}, \\ z &= \bar{\xi} + i\bar{\eta} = -\bar{\epsilon} a + 2\bar{\epsilon} \frac{a + ib\bar{\tau}}{1 + \bar{\tau}^2}. \end{aligned}$$

To simplify the expressions on the right hand of these equations and at the same time to bring them to well-known forms we change our parameter by means of a linear transformation,

$$\tau = i\,\frac{\tau' - 1}{\tau' + 1}.$$

Effecting this change in (4′) we have

$$(5) \quad \begin{aligned} w &= \frac{\epsilon}{2}\left[(a - b)\tau' + \frac{a + b}{\tau'} \right], \\ z &= \frac{\bar{\epsilon}}{2}\left[(a + b)\bar{\tau}' + \frac{a - b}{\bar{\tau}'} \right]. \end{aligned}$$

The function $\frac{1}{2}[\zeta + (1/\zeta)]$ is one of the most common in function theory. The Riemann surface belonging to it is two-sheeted, the two sheets being connected along the branch-cut running from -1 to $+1$. To concentric circles about the origin in the ζ-plane correspond confocal ellipses on the surface. To the pencil of straight lines through the origin correspond hyperbolas confocal with the above-mentioned ellipses. A single ellipse lies entirely in one sheet, but the branches of a single hyperbola lie half in one sheet and half in the other. The Riemann surfaces of w and z defined by (5) differ from this surface in no essential respect.[1]

However, for some further simplification, and for uniformity with cases to be discussed later, we make a second change of parameter. We may write

$$(a - b)\tau' + \frac{a + b}{\tau'} = (a - b)\tau' + \frac{a^2 - b^2}{(a - b)\tau'} = \tau_1 + \frac{c^2}{\tau_1},$$

$$\tau_1 = (a - b)\tau'.$$

[1] See Lewent, Konforme Abbildung, p. 63.

Where $c^2 = a^2 - b^2$, c being the focal distance. Thus we have

(6)
$$w = \frac{\epsilon}{2}\left[\tau_1 + \frac{c^2}{\tau_1}\right],$$

where

$$\tau_1 = (a - b)\tau';$$

and

$$\bar{z} = \frac{\bar{\epsilon}}{2}\left[\bar{\tau}_2 + \frac{c^2}{\bar{\tau}_2}\right],$$

where

$$\tau_2 = (a + b)\tau'.$$

Certainly the simplest complex configuration is the real straight line, and the simplest position of the real straight line is in coincidence with the ξ-axis, $\eta = 0$. The picture of the line in this position is given by the ensemble of point-pairs connected by the relation $w = \bar{z}$ by virtue of $\eta = (w - \bar{z})/2i$. It is known in Conformal Geometry[1] that any real analytic curve is conformally transformable into a straight line, in particular, into the ξ-axis (small regions about a point of each curve being taken). Conversely the ξ-axis may be transformed conformally to any analytic curve. The transformation effecting this, operating on $w = \bar{z}$ gives $w = f(\bar{z})$ or more generally

$$F(\bar{z}, w) = 0.$$

For a small region about a point ξ' on the ξ-axis the points symmetric (in the Schwartzian sense) to each other with respect to the ξ-axis go over into points symmetric with respect to the arc of the analytic curve through the transform of ξ'. We shall give this principle prominent place in this paper. And, since our transformations are of the simplest sort, consisting only of translations, rotations and inversions, the principle just stated holds *in the large*, that is, throughout the finite portion of the plane.

The parameter τ is complex and we ascertain first the correspondence between the τ-plane and the τ'-plane, the two variables being connected by the relation:

$$\tau = i\,\frac{\tau' - 1}{\tau' + 1}.$$

[1] Kasner, "Conformal Geometry," International Congress of Mathematicians, Cambridge, 1912.

Solving for τ' and calling for the moment $\tau = x + iy$ and $\tau' = x' + iy'$ we have, following the usual method of Holzmüller,[1]

$$x' + iy' = -\frac{x + i(y+1)}{x + i(y-1)},$$

$$x' - iy' = -\frac{x - i(y+1)}{x - i(y-1)},$$

and

$$x'^2 + y'^2 = \frac{x^2 + (y+1)^2}{x^2 + (y-1)^2} = \frac{p^2}{q^2} = r^2.$$

Where p and q are radii vectores from the points $-i$ and $+i$. The variation of p and q gives the pencil of circles with $\pm i$ as limiting points. The real axis of τ is given by $p = q$ and since $r = p/q = 1$, we see that to the real axis of τ corresponds the unit circle of τ'. Further, to circles of the pencil in the upper half of the τ-plane, $p > q$ and hence $r = p/q > 1$ correspond concentric circles about the origin of the τ'-plane outside the unit circle, that is, the upper half of the τ-plane maps into the outside of the unit circle, and consequently the lower half of the τ-plane maps into the region inside the unit circle of the τ'-plane. It is easy also to show that the imaginary axis of the τ-plane goes over into the real axis of the τ'-plane.

Returning now to equations (6) let us write

(7)
$$\left.\begin{aligned} w &= \epsilon\sigma_1 \\ \text{and}\quad z &= \bar{\epsilon}\,\bar{\sigma}_2 \end{aligned}\right\},$$

$$\text{where}\quad \left\{\begin{aligned} \sigma_1 &= \frac{1}{2}\left(\tau_1 + \frac{c^2}{\tau_1}\right), \qquad \tau_1 = (a-b)\tau'; \\ \sigma_2 &= \frac{1}{2}\left(\tau_2 + \frac{c^2}{\tau_2}\right), \qquad \tau_2 = (a+b)\tau'. \end{aligned}\right.$$

Let $\sigma_k = s_k' + is_k''$ and $\tau_k = t_k' + it_k''$; $(k = 1, 2)$. Separating the last equations of (7) into their real and pure imaginary parts we have

(8)
$$s_k' = \frac{t_k'}{2}\left[\frac{c^2}{t_k'^2 + t_k''^2} + 1\right]$$

and

[1] "Theorie der Isogonalen Verwandschaften."

$$s_k'' = -\frac{t_k''}{2}\left[\frac{c^2}{t_k'^2 + t_k''^2} - 1\right].$$

To circles about the origin in the τ_k-plane, $t_k'^2 + t_k''^2 = r^2$, correspond ellipses whose equations are

$$(9) \qquad \frac{4s_k'^2}{\left(\frac{c^2}{r} + r\right)^2} + \frac{4s_k''^2}{\left(\frac{c^2}{r} - r\right)^2} = 1.$$

To lines through the origin of the τ_k-plane, $t_k'' = mt_k'$, correspond hyperbolas whose equations are

$$(10) \qquad \frac{s_k'^2}{\frac{c^2}{1 + m^2}} - \frac{s_k''^2}{\frac{c^2m^2}{1 + m^2}} = 1.$$

These ellipses and hyperbolas are confocal for

$$\frac{1}{4}\left(\frac{c^2}{r} + r\right)^2 - \frac{1}{4}\left(\frac{c^2}{r} - r\right)^2 = c^2$$

and

$$\frac{c^2}{1 + m^2} + \frac{c^2m^2}{1 + m^2} = c^2,$$

c being the focal distance.

If equations (9) and (10) be expressed in terms of $a - b$ and $a + b$, that is, if σ_1 and σ_2 be expressed as functions of τ' the results are, writing a' for $a + b$ and b' for $a - b$,

$$(9') \qquad \frac{4s_1'^2}{\left(\frac{a'}{r} + b'r\right)^2} + \frac{4s_1''^2}{\left(\frac{a'}{r} - b'r\right)^2} = 1, \qquad \frac{4s_2'^2}{\left(\frac{b'}{r} + a'r\right)^2} + \frac{4s_2''^2}{\left(\frac{b'}{r} - a'r\right)^2} = 1,$$

$$(10') \qquad \frac{s_k'^2}{\frac{4a'b'}{1 + m^2}} - \frac{s_k''^2}{\frac{4a'b'm^2}{1 + m^2}} = 1.$$

These equations show that to any circle of radius r in the τ'-plane in general correspond different ellipses in the σ_1- and

σ_2-surfaces. There is one exception, namely, $r = 1$. In this case each of the equations (9') becomes

$$\frac{s'^2}{a^2} + \frac{s''^2}{b^2} = 1. \tag{11}$$

To any line in the τ'-plane through the origin correspond the same hyperbolas, but we are not to understand that as τ' de-

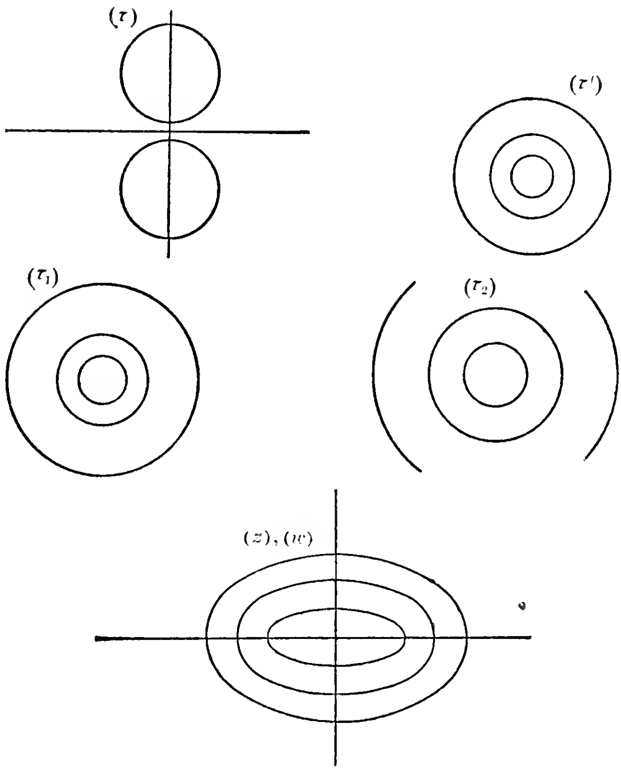

FIG. 1.

scribes any given line in its plane σ_1 and σ_2 describe the branches of the corresponding hyperbola together. A little examination shows that they never move in the same direction on the hyperbola but always in opposite directions, passing each other on the common ellipse (11) of the two planes. We do not carry this investigation beyond this point. Further details are mentioned by Study, pp. 98–103.

If our ellipse has a real branch, that is, if $\epsilon = 1$ the w- and z-surfaces are identical with the σ_1- and $\bar{\sigma}_2$-surfaces. If $\epsilon = i$, the w- and z-surfaces are got by rotating the σ_1- and $\bar{\sigma}_2$-surfaces through 90° and − 90° respectively.

We have shown in this section that the real axis of the parameter plane (τ) maps into the real branch of the ellipse $\epsilon = 1$, while for the ellipse $\epsilon = i$ the real axis of τ maps into the real branch of the ellipse $\epsilon = 1$ turned through 90°. The picture-points z and w are united in position for the ellipse $\epsilon = 1$, and are diametrically opposite for the ellipse $\epsilon = i$. The elliptic pencil of circles with limiting points $\pm i$ have been shown to map into confocal ellipses, circles symmetric with respect to the real axis in the τ-plane go into ellipses symmetric with respect to the real branch of the ellipse $\epsilon = 1$, and for the case $\epsilon = i$ symmetric with respect to the ellipse $\epsilon = 1$ turned through 90°. It may be shown that the hyperbolic pencil orthogonal to the elliptic pencil maps into the confocal hyperbolas.[1] Corresponding curves and points are shown in the figures.

5. *The Real Hyperbola.*—

$$\frac{\xi^2}{a^2} - \frac{\eta^2}{b^2} = 1, \qquad a > b.$$

Introducing the parameter in the same way as in the case of the ellipse we have:

$$\xi = a\frac{1 + \tau^2}{1 - \tau^2} = - a + \frac{2a}{1 - \tau^2},$$

$$\eta = \frac{2b\tau}{1 - \tau^2}.$$

Hence

$$w = - a + 2\frac{a + ib\tau}{1 - \tau^2},$$

$$z = - a + 2\frac{a + ib\bar{\tau}}{1 - \bar{\tau}^2}.$$

Changing the parameter by means of the linear transformation:

$$\tau = \frac{\tau' - 1}{\tau' + 1},$$

[1] Holzmüller, p. 63.

we have,

(1)
$$w = \frac{1}{2}\left[(a + ib)\tau' + \frac{a - ib}{\tau'}\right],$$
$$z = \frac{1}{2}\left[(a + ib)\bar{\tau}' + \frac{a - ib}{\bar{\tau}'}\right].$$

Or, as in the case of the ellipse, if we put

(2) $\tau_1 = (a + ib)\tau'$ and $\tau_2 = (a - ib)\tau'$, and $a^2 + b^2 = c^2$,

c being the focal distance, we have,

(3)
$$w = \frac{1}{2}\left[\tau_1 + \frac{c^2}{\tau_1}\right],$$
$$z = \frac{1}{2}\left[\bar{\tau}_2 + \frac{c^2}{\bar{\tau}_2}\right].$$

When the equations of (3) are resolved into their real and pure imaginary parts we have, as in the preceding case:

(4)
$$u = \frac{t_1'}{2}\left(\frac{c^2}{t_1'^2 + t_1''^2} + 1\right),$$
$$v = -\frac{t_1''}{2}\left(\frac{c^2}{t_1'^2 + t_1''^2} - 1\right);$$
$$x = \frac{t_2'}{2}\left(\frac{c^2}{t_2'^2 + t_2''^2} + 1\right),$$
$$y = -\frac{t_2''}{2}\left(\frac{c^2}{t_2'^2 + t_2''^2} - 1\right).$$

Since $\tau_1 = (a + ib)\tau'$ and $\tau_2 = (a - ib)\tau'$, the configuration of the τ'-plane consisting of concentric circles and radial lines, is not altered, lines and circles merely going over into lines and circles. In particular the real axis of τ' goes into the line with slope b/a in the τ_1-plane, and the line with slope $-(b/a)$ in the τ_2-plane. Since the expansion $|a + ib|$ is the same for both τ_1 and τ_2, a circle of τ' goes into equal circles in the τ_1- and τ_2-planes. To concentric circles and radial lines of the τ_1- and τ_2-planes there correspond ellipses and hyperbolas in the w-

and z-surfaces. Their equations are:

$$\frac{4u^2}{\left(\frac{c^2}{r}+r\right)^2}+\frac{4v^2}{\left(\frac{c^2}{r}-r\right)^2}=1, \qquad \frac{u^2}{\frac{c^2}{1+m_1^2}}-\frac{v^2}{\frac{c^2m_1^2}{1+m_1^2}}=1;$$

and

$$\frac{4x^2}{\left(\frac{c^2}{r}+r\right)^2}+\frac{4y^2}{\left(\frac{c^2}{r}-r\right)^2}=1, \qquad \frac{x^2}{\frac{c^2}{1+m_2^2}}-\frac{y^2}{\frac{c^2m_2^2}{1+m_2^2}}=1,$$

where

$$\tan^{-1} m_1 = \tan^{-1} m + \tan^{-1}\frac{b}{a} : \tan^{-1} m_2 = \tan^{-1} m - \tan^{-1}\frac{b}{a},$$

m being the slope of a line through the origin in the τ'-plane. The ellipses correspond, but the hyperbolas are, in general, different, the exception being where $m_1 = b/a$ and $m_2 = -b/a$. For this case the above equations reduce to:

$$\frac{u^2}{a^2}-\frac{v^2}{b^2}=1, \qquad \frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Now the transformation $\tau = (\tau' - 1)/(\tau' + 1)$ converts the real axis of τ into the real axis of τ', and as we noted the lines of slope b/a and $-b/a$ in the τ_1- and τ_2-planes correspond to the real axis of τ' and hence to the real axis of τ. So the real axis of τ maps into the real branch of the hyperbola.

It can be shown[1] that to concentric circles about the origin in (τ') correspond the pencil of circles having ± 1 for their limiting points, and to the radial lines in the τ'-plane correspond the pencil of circles with vertices ± 1. So we have the case corresponding to that of the ellipse: the hyperbolic pencil of circles in the τ-plane with the common points ± 1 maps into confocal hyperbolas in the w- and z-surfaces; in particular, the radical axis of the pencil going into the real branch of the hyperbola. The orthogonal pencil with ± 1 as limiting points goes into the confocal ellipses.

6. *The Real Parabola.*—We take for the equation of the parabola

(1) $$\eta^2 = 4p(\xi + p),$$

[1] See Holzmüller, p. 63.

with the finite focus at the origin. Form the pencil,

$$\eta = 2\tau(\xi + p).$$

Then,

$$\xi = p\frac{1 - \tau^2}{\tau^2}, \qquad \xi = -p + \frac{p}{\tau^2}, \qquad \eta = \frac{2p}{\tau},$$

and,

$$w = -p + p\frac{1 + 2i\tau}{\tau^2}, \qquad z = -p + p\frac{1 + 2i\bar{\tau}}{\bar{\tau}^2}.$$

As in the previous cases we simplify these expressions by a linear transformation of the parameter

(2) $$\tau = \frac{1}{\tau_1 - i} \text{ for } w, \text{ and } \tau = \frac{1}{\tau_2 + i} \text{ for } z.$$

Thus we get

(3) $$w = p\tau_1^2, \qquad z = p\bar{\tau}_2^2.$$

The Riemann surface belonging to $p\tau^2$ consists of two sheets joined along the positive real axis. To lines parallel to the axis of reals $t_1'' =$ const. correspond confocal parabolas,

$$v^2 = 4pt_1''^2(u + pt_1''^2).$$

To lines parallel to the imaginary axis, $t_1' =$ const., correspond the orthogonal trajectories of the preceding set, namely,

$$v^2 = -4pt_1'^2(u - pt_1'^2).$$

These two families of parabolas are confocal, having the origin for their finite focus.

We get similar equations for z,

$$y^2 = 4pt_2''^2(x + pt_2''^2),$$

$$y^2 = -4pt_2'^2(x - pt_2'^2).$$

Solving (2) for τ_1 and τ_2 we may write:

$$\tau_k = \frac{1 \pm i\tau}{\tau}, \qquad (k = 1, 2)$$

the upper sign going with τ_1 and the lower with τ_2. Separating this equation into its real and pure imaginary parts we have

$$t_k' = \frac{t'}{t'^2 + t''^2}, \qquad t_k'' = \pm \frac{t'^2 + t''^2 \mp t''}{t'^2 + t''^2}.$$

To lines parallel to the real axis of τ_k, $t_k'' = b$, correspond a parabolic pencil of circles through the origin with centers on the imaginary axis, $(b \mp 1)(t'^2 + t''^2) \pm t'' = 0$. In particular to the lines $t_k'' = \pm 1$ corresponds $t'' = 0$, the real axis of τ. Again, to lines parallel to the imaginary axis, $t_k' = a$, corresponds a second parabolic pencil of circles through the origin with centers on the real axis, the orthogonal trajectories of the preceding pencil. The equation of this last pencil is $a(t'^2 + t''^2) - t' = 0$.

To the real axis of τ then corresponds in the w- and z-surfaces the curves $v^2 = 4p(u + p)$ and $y^2 = 4p(x + p)$, that is, the real branch of our parabola. As τ describes a circle of radius r, in its upper half-plane, say, $\bar{\tau}$ describes a congruent circle in the lower half-plane. Correspondingly τ_1 describes a straight line $t_1'' = (1/2r) + 1$ in its plane and τ_2 describes a straight line $t_2'' = (1/2r) - 1$, and in turn w and z describe the confocal parabolas:

$$v^2 = 4p\left(\frac{1}{2r} + 1\right)^2 \left[u + p\left(\frac{1}{2r} + 1\right)^2\right],$$

$$y^2 = 4p\left(\frac{1}{2r} - 1\right)^2 \left[x + p\left(\frac{1}{2r} - 1\right)^2\right].$$

As τ describes a circle of radius r' of the orthogonal pencil, $\bar{\tau}$ describes the same circle and w and z describe the same parabola

$$v^2 = -\frac{p}{r'^2}\left(u - \frac{p}{4r'^2}\right),$$

$$y^2 = -\frac{p}{r'^2}\left(x - \frac{p}{4r'^2}\right).$$

Both families of parabolas are double decked, that is, both sheets of the Riemann surface are filled with parabolas. Those extending infinitely to the right have both branches in the same sheet. Those extending infinitely to the left change sheets over the branch-cut running from the origin to infinity positively.

CHAPTER II.

THE COMPLEX CONIC, PRELIMINARY CONSIDERATIONS.

While the purpose of this paper is the reduction of the equation and the real representation of the imaginary conic, yet we shall find it advantageous to spend some time with the more elementary configurations. As a result should come a proper orientation in the field of "geometry in the domain of the complex"; also suggestions as to methods of procedure in the question proper. Accordingly we give our attention first to

The Complex Line.

7. *Relation of the Pictures of a Complex Line and Its Conjugate, or the Group Property of the Transformations Belonging to the Complex Line and Its Conjugate.*—The general equation of the complex line is

$$\Lambda : \alpha\xi + \beta\eta + \gamma = 0,$$

where ξ, η, α, β, γ have the form $n_1 + in_2$ and the ratios $\alpha : \beta : \gamma$ are not all real. This geometric configuration has ∞^2 complex points.[1] There are no conjugate pairs of points on the line, with one exception, namely, the real point of the line which is its own conjugate.

This fact leads us to consider in connection with Λ its conjugate,

$$\bar{\Lambda} : \bar{\alpha}\xi + \bar{\beta}\eta + \bar{\gamma} = 0.$$

$\bar{\Lambda}$ is the "locus" of the conjugate of the points of Λ, and conversely. The two loci intersect in their common, self-conjugate real point.

The corresponding reverse conformal transformations picturing the imaginary points of Λ and $\bar{\Lambda}$ are, by virtue of the relations § 2, (1)

$$(1)\quad \begin{aligned} \Lambda &: \alpha\xi + \beta\eta + \gamma = 0; & T &: \mu w + \nu\bar{z} + 2\gamma = 0, \\ \bar{\Lambda} &: \bar{\alpha}\xi + \bar{\beta}\eta + \bar{\gamma} = 0; & T' &: \bar{\nu} w + \bar{\mu}\bar{z} + 2\bar{\gamma} = 0, \end{aligned}$$

where $\mu = \alpha - i\beta$ and $\nu = \alpha + i\beta$.

[1] Hereafter the term *point* will mean *complex point.*

Let us now for the moment regard the planes (w) and (z) as coincident. Applying T to a point z' we have

(2) $$\mu w' + \nu \bar{z}' + 2\gamma = 0,$$

then applying T' to w'

(3) $$\bar{\nu} w'' + \bar{\mu}\bar{w}' + 2\bar{\gamma} = 0.$$

On elimination of w' between (3) and the conjugate of (2),

$$\bar{\mu}\bar{w}' + \bar{\nu} z' + 2\bar{\gamma} = 0,$$

there results

$$w'' = z'.$$

The result is the same if we reverse the order of application. Hence the theorem:

The reverse conformal transformations belonging to a line and its conjugate applied consecutively leave the points of the plane in place:

$$T' = T^{-1}, \qquad TT' = T'T = 1.$$

In case the line is real $\alpha = a$, $\beta = b$, $\gamma = c$, $\mu = a - ib$, $\nu = a + ib = \bar{\mu}$ then $T = T' : \mu w + \bar{\mu}\bar{z} + 2c = 0$, that is, the real line is its own conjugate and further $TT' = T^2 = 1$, the transformation belonging to it is involutorial.

8. *Reduction of the Equation of the Complex Line to Canonical Form.*—Putting Λ and $\bar{\Lambda}$ in the form $R + iT$, R, T being linear functions in ξ, η with real coefficients, we have

$$\Lambda, \bar{\Lambda} : L_1 \pm iL_2 : a_1\xi + b_1\eta + c_1 \pm i(a_2\xi + b_2\eta + c_2) = 0.$$

Thus Λ and $\bar{\Lambda}$ are identified as members of a complex pencil

$$\Pi : L_1 + \kappa L_2 = 0,$$

where $\kappa = k' + ik''$. Belonging to this pencil there is of course a single infinity of real lines, k' arbitrary, k'' zero. On each value of k', k_i' say, there is built up a single infinity of imaginary lines, k_i'' arbitrary. We may thus distribute the double infinity of complex lines into a single infinity of *sub-pencils* each with real bases and each containing two real lines, the bases, $k' = k_i'$, and ∞, and a single infinity of imaginary lines, k'' arbitrary. According to this classification Λ and $\bar{\Lambda}$ belong to the sub-pencil $k' = 0$ with bases L_1 and L_2.

We proceed now to simplify the equations of Λ and $\bar{\Lambda}$ by referring them to other bases, namely a certain rectangular pair. Assuming $a_1b_2 - a_2b_1 \neq 0$ and $a_1a_2 + b_1b_2 \neq 0$[1] Λ and $\bar{\Lambda}$ have their common real point finitely located. We translate the origin to this point and at the same time write Λ in the so-called normal form of elementary geometry,

$$\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\xi + \frac{\beta}{\sqrt{\alpha^2+\beta^2}}\eta = 0$$

or

$$\xi \cos\theta + \eta \sin\theta = 0$$

where

$$\cos\theta = \cos(s + it) = \cosh t \cos s - i \sinh t \sin s,$$

and

$$\sin\theta = \sin(s + it) = \cosh t \sin s + i \sinh t \cos s.$$

Thus the normal forms of Λ and $\bar{\Lambda}$ are

(1) $$\xi \cos s + \eta \sin s \pm i \tanh t(\xi \sin s - \eta \cos s) = 0.$$

Functions of the angle θ, or of its component parts are furnished by the relations

$$\tan s = \frac{b_1 + b_2 \tan q}{a_1 + a_2 \tan q} = -\frac{a_1 \tan q - a_2}{b_1 \tan q - b_2}$$

and

$$\tanh t = \frac{a_1 - a_2 \cot q}{b_1 - b_2 \tan q} = -\frac{b_1 + b_2 \cot q}{a_1 + a_2 \tan q},$$

where $q = \frac{1}{2}\operatorname{arc}(\alpha^2 + \beta^2)$. The sign of $\sqrt{\alpha^2+\beta^2}$ is that required when α and β are real.

Equation (1) shows Λ and $\bar{\Lambda}$ referred to new bases, two perpendicular lines belonging to the pencil Π. These new bases are connected by the relation:[2]

$$\Pi : L' \cosh t + \lambda L'' \sinh t \equiv (1 + \lambda)(L_1 + \kappa L_2),$$

where

$$L' = \xi \cos s + \eta \sin s,$$

$$L'' = \xi \sin s - \eta \cos s.$$

Taking L' and L'' as coordinate axes our equation of the com-

[1] If $a_1a_2 + b_1b_2 = 0$, L_1 and L_2 are perpendicular to each other and the second transformation worked out above is not necessary.

[2] Newenglowski, "Cours de Géométrie Analytique," sec. 446, old ed.

plex line reduces to the form:[1]

$$\Lambda, \bar{\Lambda} : \eta = \pm i \tanh t \xi.$$

9. *Real Representation of the Complex Line.*—We have thus reduced the equation of the complex line containing two effective complex coefficients to one in which there is but a single coefficient and it is pure imaginary in form. Writing Λ and $\bar{\Lambda}$ separately with their accompanying transformations we have

$$\Lambda : \eta = i \tanh t \xi; \qquad T : w = e^{-2t}\bar{z},$$
$$\bar{\Lambda} : \eta = - i \tanh t \xi; \quad T' : w = e^{2t}\bar{z}.$$

The limits of t and $\tanh t$ are $-\infty < t < +\infty$, $-1 < \tanh t < +1$. If we allow t to become infinite and assume its upper and lower bounds then we have the minimal pair of the pencil,

$$\xi + i\eta = 0,$$
$$\xi - i\eta = 0.$$

Since e^{2t} and e^{-2t} are real the picture-pairs z and w are observed to lie on rays symmetric with respect to the first bisector of Λ and $\bar{\Lambda}$.

If we introduce with Study, page 52, a parameter ρ measuring the distance from the vertex of the pencil to the points on Λ we have

$$(2) \qquad \begin{aligned} \xi &= \rho \cos it = \rho \cosh t, \qquad & w &= \frac{1}{e^t}\rho, \\ \eta &= \rho \sin it = i\rho \sinh t, \qquad & z &= e^t\bar{\rho}. \end{aligned}$$

Since $wz = |\rho|^2$, for any ρ, w and z are seen to be inverse points with respect to the circle of radius $|\rho|$, w having the same argument as ρ, and z its negative.

The Complex Circle.

In a manner similar to that used in the case of the line let us take up the case of the circle.

10. *Group Property of Inversion with Respect to a Complex Circle and its Conjugate.*—The equation of a complex circle and

[1] Cf. Study, pp. 29, 52.

its conjugate with their accompanying transformations are

$$\mathrm{K} : \xi^2 + \eta^2 + 2\alpha\xi + 2\beta\eta + \gamma = 0,$$
$$T : w\bar{z} + \mu w + \nu\bar{z} + \gamma = 0,$$
$$\bar{\mathrm{K}} : \xi^2 + \eta^2 + 2\bar{\alpha}\xi + 2\bar{\beta}\eta + \bar{\gamma} = 0,$$
$$T' : w\bar{z} + \bar{\nu}w + \bar{\mu}\bar{z} + \bar{\gamma} = 0,$$

where ξ, η, α, $\cdots$ are complex quantities and $\mu = \alpha - i\beta$, $\nu = \alpha + i\beta$. Neither T nor T' is involutoric, but, as before, applying T to a point z' of (z)

(1) $$w'\bar{z}' + \mu w' + \nu\bar{z}' + \gamma = 0;$$

and applying T' to w' of (w)

(2) $$w''\bar{w}' + \bar{\nu}w'' + \bar{\mu}\bar{w}' + \bar{\gamma} = 0,$$

then subtracting from (2) the conjugate of (1) we get

$$(\bar{w}' + \bar{\nu})(w'' - z') = 0.$$

Obviously w' cannot equal $-\nu$ for all values of z', hence $w'' = z'$ and $T' = T^{-1}$, $TT' = 1$.

In this case and in the preceding we might have expressed w explicitly in terms of z and arrived at the same result by substitution. This relation is also evident by a mere examination of the two expressions T and T', considering in the one z as the independent variable and in the other w as the independent variable. When $\alpha = a$, $\beta = b$ and $\gamma = c$ that is, when the circle is real (in the sense of Segre) we have $\mu = a - ib$ and $\nu = a + ib = \bar{\mu}$, so $T \equiv T' : w\bar{z} + \mu w + \bar{\mu}\bar{z} + c = 0$.

11. *Reduction of Equations of* K *and* $\bar{\mathrm{K}}$ *to Canonical Form.*—Breaking up K and $\bar{\mathrm{K}}$ into their real and pure imaginary parts we have

$$\mathrm{K}, \bar{\mathrm{K}} : \xi^2 + \eta^2 + 2a_1\xi + 2b_1\eta + c_1 \pm i(2a_2\xi + 2b_2\eta + c_2) = 0.$$

They are thus seen to be members of a complex pencil

$$\mathrm{H} : C_1 + \kappa R = 0,$$

where $C_1 \equiv \xi^2 + \eta^2 + 2a_1\xi + 2b_1\eta + c_1 = 0$ is a real circle of the pencil, $R \equiv 2a_2\xi + 2b_2\eta + c_2 = 0$ is the radical axis of the pencil (also real), and $\kappa = k' + ik''$.

Clearly we may simplify our configuration by a change of axes. This we do making the axis of centers the ξ-axis and the radical axis the η-axis. As a result we have

$$\Pi : \xi^2 + \eta^2 + 2a'\xi + c' + 2\lambda\xi = 0,$$

where

$$a' = \frac{2(a_1a_2 + b_1b_2) - c_2}{2\sqrt{a_2^2 + b_2^2}}, \qquad c' = C_1(\xi_0, \eta_0)$$

and

$$\lambda = \sqrt{a_2^2 + b_2^2}\kappa; \qquad (\xi_0, \eta_0),$$

coordinates of the new origin.

Among the real circles of Π there is one with its center at the (new) origin and is given by $\lambda = -2a'$. Taking this circle with the radical axis as bases we have

$$\Pi : \xi^2 + \eta^2 + 2\mu\xi + c' = 0,$$

where $\mu = \lambda + 2a'$. The character of the pencil—whether hyperbolic, elliptic or parabolic—depends on whether c' is greater than, less than, or equal zero.

Our original circles, K and $\bar{\text{K}}$ referred to the new bases are seen to be given by the values $\mu = a' \pm i\sqrt{a_2^2 + b_2^2}$, that is,

$$\mu = m' + im'', \qquad m' = \frac{2(a_1a_2 + b_1b_2) - c_2}{2\sqrt{a_2^2 + b_2^2}}, \qquad m'' = \sqrt{a_2^2 + b_2^2}.$$

We have thus succeeded in reducing the equation of the complex circle

$$\text{K} : \xi^2 + \eta^2 + 2\alpha\xi + 2\beta\eta + \gamma = 0,$$

containing three complex coefficients to a canonical form

$$\text{K} : \xi^2 + \eta^2 + 2\mu\xi + c = 0,$$

containing but a single complex coefficient.

12. The transformations belonging to K and $\bar{K}$ expressed in the canonical form are

$$\text{(3)} \qquad \begin{aligned} \text{K} = 0; &\qquad T : w\bar{z} + \mu(w + \bar{z}) + c = 0, \\ \bar{\text{K}} = 0; &\qquad T' : w\bar{z} + \bar{\mu}(w + \bar{z}) + c = 0. \end{aligned}$$

These transformations of course are not involutoric any more than the original ones were. So we cannot consider (z) and (w) coincident. To study the corresponding movements in (z) and (w) we introduce a parameter plane defined in the following manner: Setting

$$\eta = 0, \quad \cdots, \quad w = \bar{z} \text{ in } (3),$$

then

$$\xi^2 + 2\mu\xi + c = 0, \quad \cdots, \quad w^2 + 2\mu w + c = 0$$

and

$$\xi', \xi'' = -\mu \pm \sqrt{\mu^2 - c} = w', w'' \text{ or } \bar{z}', \bar{z}''.$$

Form the pencil of lines $\eta = \tau(\xi - \xi'')$, $\tau = s + it$. Hence

$$\xi - \xi'' = \frac{2\rho}{1 + \tau^2}, \qquad w - w'' = \frac{2\rho}{1 - i\tau},$$

and

$$\eta = \frac{2\rho\tau}{1 + \tau^2}, \qquad z - z'' = \frac{2\bar{\rho}}{1 - i\bar{\tau}},$$

where $\rho = \sqrt{\mu^2 - c} = r_1 + ir_2$, the radius of the complex circle. Setting $w' = w - w''$, $z' = z - z''$ and dropping the primes,

$$(4) \qquad w = \frac{2\rho}{1 - i\tau} \quad \text{and} \quad z = \frac{2\bar{\rho}}{1 - i\bar{\tau}}.$$

If τ describe its axis of reals, $t = 0$, we have, on elimination of s,

$$(u - r_1)^2 + (v - r_2)^2 = r^2$$

and

$$(x - r_1)^2 + (y - r_2)^2 = r^2,$$

where $r^2 = |\rho|^2 = r_1^2 + r_2^2$. w and z thus are seen to describe congruent circles in their respective planes.

If τ describe its axis of pure imaginaries, $s = 0$, there result on the elimination of t

$$v = \frac{r_2}{r_1} u, \qquad \text{and} \qquad y = -\frac{r_2}{r_1} x.$$

Our picture planes are thus seen to be divided into four regions corresponding to the four quadrants of the parameter plane. The paired points of the two congruent circles in (z) and (w) are the points which in the real case are united in position picturing the ∞^1 real points of the circle. We superpose the planes now and notice a sort of *unfolding* process of the picture planes due to the entrance of the complex quantity. Let us examine the parabolic case. Here we have:

$$\xi^2 + \eta^2 + 2\mu\xi = 0,$$

$$\xi' = 0, \qquad \xi'' = -2\mu, \qquad \rho = \mu = r_1 + ir_2.$$

Considering the planes superposed, the origins coincident and the axes of reals of (w) and (z) coinciding with the ξ-axis of the cartesian plane we have:

$$w = -2\mu + \frac{2\mu}{1 - i\tau} = \frac{2i\mu\tau}{1 - i\tau},$$

$$z = -2\bar{\mu} + \frac{2\bar{\mu}}{1 - i\bar{\tau}} = \frac{2i\bar{\mu}\bar{\tau}}{1 - i\bar{\tau}}.$$

If

$$\begin{array}{llllll} \tau = -\infty, & -1, & 0, & +1, & +\infty, \\ w = w_\infty, & w_{-1}, & 0, & w_{+1}, & w_\infty, \\ z = z_\infty, & z_{-1}, & 0, & z_{+1}, & z_\infty. \end{array}$$

If

$$\begin{array}{lllllll} \tau = -\infty, & -i, & -\frac{1}{2}i, & 0, & +\frac{1}{2}i, & +i, & +\infty, \\ w = w_\infty, & \infty, & w_{\frac{1}{2}}, & 0, & -\frac{2}{3}\mu, & -\mu, & w_\infty, \\ z = z_\infty, & -\mu, & -\frac{2}{3}\mu, & 0, & z_{\frac{1}{2}}, & \infty, & z_\infty. \end{array}$$

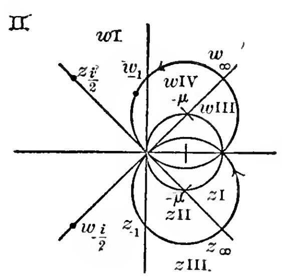

FIG. 2.

Thus having in mind that $\tau \rightarrow w$ is a direct transformation and $\tau \rightarrow z$ is a reverse transformation, the angles being preserved in sense in the one and reversed in the other, we shall see just what regions of (z) and (w) correspond. They are so indicated in the figure. The situation is apparent if we consider μ as a sort of parameter varying in its pure imaginary part only. Let $\mu = m_1 + im_2$. $m_2 = 0$ gives a real circle of the pencil with its center $(-m_1, 0)$. The variation of m_2 from 0 is seen to cause an *unfolding* of the *pseudo-real* pair from the real circle, the centers of these circles being $(-m_1, \pm m_2)$. For m_2 negative merely interchanges the figures z and w. For m_1 negative, the figures are reflected over the η-axis. It is interesting to trace the *path-curves* of z and w due to the variation of m_2, holding τ fixed. They are straight lines as is evident when we write

$$w = \frac{2i\tau}{1 - i\tau}(m_1 + im_2) \qquad \text{and} \qquad z = \frac{2i\bar{\tau}}{1 - i\bar{\tau}}(m_1 - im_2).$$

In particular the points z, w, whose united positions represent the ∞^1 real points in the real case trace the lines

$$y = s(x + 2m_1) \qquad \text{and} \qquad v = s(u - 2m_1).$$

The hyperbolic and elliptic cases do not introduce any essential differences.

Study has given the elementary transformations making up reflection with respect to the complex circle.[1] In the light of the foregoing considerations we may state the following set, which is equivalent to that given by Study. Calling the congruent circles in (z) and (w) picturing the real axis of the parameter, the Congruent-Pair, and the lines picturing the axis of pure imaginaries, the Basic-Lines, the transformations in question are as follows:

The transformation picturing the complex points on an imaginary circle consists of three elementary reflections, namely, a reflection over the center axis determined by the circle and its conjugate, a reflection over one of the Basic-Lines, and a reflection over one of circles of the Congruent-Pair. These may be taken in any order but the Basic-Line and the circle of the Congruent-Pair must not belong to the same planes.

[1] Page 32.

CHAPTER III.

THE COMPLEX CONIC, REDUCTION OF EQUATION.

With the processes and results of the preliminary considerations in mind we give our attention now to the subject proper.

13. *The General Equation and Its Corresponding Transformation.*—The general equation of a complex conic and its conjugate are,

$$\text{(I)}\qquad \begin{aligned} \Gamma &: \alpha\xi^2 + 2\beta\xi\eta + \gamma\eta^2 + 2\delta\xi + 2\epsilon\eta + \zeta = 0, \\ \bar\Gamma &: \bar\alpha\xi^2 + 2\bar\beta\xi\eta + \bar\gamma\eta^2 + 2\bar\delta\xi + 2\bar\epsilon\eta + \bar\zeta = 0, \end{aligned}$$

where all the quantities entering are complex and the equations are irreducible.

Each configuration contains a double infinity of points and the conjugates of the points of the one lie on the other. They have four points in common. These may all be real or all imaginary, or any of the intermediate cases. It is not necessary to make a separate discussion for any particular case.

The transformations picturing the complex points on these "curves" are given through the relation § 2, (1).

Thus we have

$$\text{(II)}\qquad \begin{gathered} \Gamma = 0; \\ T : \alpha_1 w^2 + 2\beta_1 w\bar z + \gamma_1\bar z^2 + 2\delta_1 w + 2\epsilon_1\bar z + 4\zeta = 0, \\ \bar\Gamma = 0; \\ T' : \bar\gamma_1 w^2 + 2\bar\beta_1 w\bar z + \bar\alpha_1\bar z^2 + 2\bar\epsilon_1 w + 2\bar\delta_1\bar z + 4\bar\zeta = 0, \end{gathered}$$

where

$$\begin{aligned} \alpha_1 &= \alpha - \gamma - 2i\beta, \\ \gamma_1 &= \alpha - \gamma + 2i\beta, \end{aligned} \qquad \beta_1 = \alpha + \gamma, \qquad \begin{aligned} \delta_1 &= 2(\delta - i\epsilon), \\ \epsilon_1 &= 2(\delta + i\epsilon). \end{aligned}$$

Each transformation is (2, 2)-determined and hence requires a two-sheeted Riemann surface for unique determination.

We note that when $\beta_1 = 0$ the variables are rationally separable,

$$\text{(III)}\qquad \alpha_1 w^2 + 2\delta_1 w = -(\gamma_1\bar z^2 + 2\epsilon_1\bar z + 4\zeta).$$

This requires that $\alpha + \gamma = 0$ which corresponds to the real case for an equilateral hyperbola. We give this case special consideration, § 18.

If we take the conjugate of T and consider z the dependent variable we observe that it is identical with T'; hence, as in the previous cases, *the transformations belonging to a complex conic and its conjugate are connected by the relation that the one is the inverse of the other*: $T' = T^{-1}$.

If the coefficients of Γ are real T and T' become identical and reduce to

$$C = 0;$$

(IV) $$T : \alpha_1 w^2 + 2b_1 w\bar{z} + \bar{\alpha}_1\bar{z}^2 + 2\delta_1 w + 2\bar{\delta}_1\bar{z} + 4h = 0.$$

The transformation $z \rightarrow w$ picturing the complex points on a real conic is an equation of the second degree in z and w; the coefficients of the square terms and the terms of first degree being conjugate complex in pairs; the coefficients of the product term and the constant term being real.[1]

If further we consider only those curves admitting ∞^1 real points, we may put

$$\xi + i\eta = z \qquad \text{and} \qquad \xi - i\eta = \bar{z}.$$

Our equation then becomes

(V) $$T : \alpha_1 z^2 + 2b_1 z\bar{z} + \bar{\alpha}_1\bar{z}^2 + 2\delta_1 z + 2\bar{\delta}_1\bar{z} + 4h = 0.$$[2]

Let us compare the expressions for the invariants and the conditions for the different species of conics in terms of the coefficients of (I) considered as real, $\alpha = a$, $\xi = x$, $\cdots$ and those of (V). We have

(I′) $$\begin{aligned} &C = 0 \cdots (\text{V}), \qquad T = 0, \\ &a + c \cdots b_1, \\ &b^2 - ac \cdots b_1 - |\alpha_1|. \end{aligned}$$

Hence in terms of the coefficients of (V) we have the following

[1] Perna, "Le Equazione delle curve in coordinate complesse coniugate," Palermo Rendiconti, 17 (1905), p. 65.

[2] Cesaro, "Sur la determination des foyers des coniques," Nouv. Ann., 60 (1901), p. 1.

conditions for the different species of proper conics and their special cases:

$$\begin{aligned} \alpha_1 &= 0, \text{ Circle,} \\ b_1 &= 0, \text{ Equilateral hyperbola,} \\ |\alpha_1| &< b_1, \text{ Ellipse,} \\ |\alpha_1| &> b_1, \text{ Hyperbola,} \\ |\alpha_1| &= b_1, \text{ Parabola.}^1 \end{aligned}$$

14. *Reduction of the General Equation to Canonical Form.*—If we write (I) in the form:

$$\text{(VI)} \quad \begin{aligned} \Gamma, \bar{\Gamma} : C_1 \pm iC_2 \equiv a_1\xi^2 + 2b_1\xi\eta + c_1\eta^2 + 2d_1\xi + 2e_1\eta \\ + f_1 \pm i(a_2\xi^2 + 2b_2\xi\eta + c_2\eta^2 + 2d_2\xi + 2e_2\eta + f_2) = 0, \end{aligned}$$

we identify Γ and $\bar{\Gamma}$ as members of a complex pencil

$$\Pi : C_1 + \kappa C_2 = 0, \qquad \kappa = k' + ik'',$$

Γ and $\bar{\Gamma}$ being given by $\kappa = i$ and $-i$ respectively.

Among the real conics, $k'' = 0$, of the pencil there is one and only one equilateral hyperbola,[2] for we have:

$$(a_1 + k'a_2)\xi^2 + 2(b_1 + k'b_2)\xi\eta + (c_1 + k'c_2)\eta^2 + \cdots = 0,$$

and the value of k' which renders this conic an equilateral hyperbola is

$$k' = -\frac{a_1 + c_1}{a_2 + c_2}.$$

If $a_1 + c_1 = 0$, C_1 is the required hyperbola and $k' = 0$ is the value of the parameter giving it. If $a_2 + c_2 = 0$, C_2 is the required curve and $k' = \infty$. If both $a_1 + c_1 = 0$ and $a_2 + c_2 = 0$ both C_1 and C_2 are equilateral hyperbolas and k' is indeterminate. In this case all the conics of the pencil are equilateral hyperbolas. (See § 18.)

Supposing the pencil to have only one equilateral hyperbola, we have

$$H : C_1 + k'C_2 = 0, \qquad k' = -\frac{a_1 + c_1}{a_2 + c_2}.$$

Taking H and C_1 as bases we have

$$\Pi : H + \lambda C_1 \equiv (1 + \lambda)(C_1 + \kappa C_2),$$

[1] Cesaro, loc. cit.
[2] Niewengloski, "Cours de Geometric Analytique," sec. 460.

where

$$\lambda = \frac{k' - \kappa}{\kappa}, \qquad k' = - \frac{a_1 + c_1}{a_2 + c_2}.$$[1]

We now take the asymptotes of the equilateral hyperbola as our coordinate axes. We have

(1) $$H : a'\xi^2 + 2b'\xi\eta - a'\eta^2 + 2d'\xi + 2e'\eta + f' = 0,$$

where

$$a' = a_1 + k'a_2, \qquad b' = b_1 + k'b_2, \qquad \cdots, \qquad k' = - \frac{a_1 + c_1}{a_2 + c_2},$$

or

$$n' = \left\{ \begin{vmatrix} n_1 & n_2 \\ a_1 & a_2 \end{vmatrix} + \begin{vmatrix} n_1 & n_2 \\ c_1 & c_2 \end{vmatrix} \right\}, \qquad n = a, b, c, \cdots, f.$$

The transformations effecting this change of coordinate axes are,

$$\xi = l\xi' - m\eta' + \xi_0,$$
$$\eta = m\xi' + l\eta' + \eta_0,$$

where

$$\xi_0 = \frac{D'}{F'}, \qquad \eta_0 = \frac{E'}{F'},$$

D', E', F' being the algebraic complements of d', e', f' in the discriminant of H,

$$l = \sqrt{\frac{-s_2}{s_1 - s_2}}, \qquad m = \sqrt{\frac{s_1}{s_1 - s_2}},$$

s_1 and s_2 being the asymptotic directions of H furnished by

$$a'\xi^2 + 2b'\xi\eta - a'\eta^2 = 0,$$

$$s_1, s_2 = \frac{-b' \pm \sqrt{F'}}{a'}.$$

As a result of this transformation the equation of H becomes

(2) $$H' : 2\xi\eta + \frac{H(\xi_0, \eta_0)}{\sqrt{F'}}, \qquad F' = a'^2 + b'^2,$$

and C_1 becomes

(3) $$C' : a\xi^2 + 2b\xi\eta + c\eta^2 + 2d\xi + 2e\eta + f = 0,$$

where

$$a = \frac{c_1 s_1 + 2b_1 - a_1 s_2}{s_1 - s_2},$$

[1] Niewenglowski, loc. cit., 446.

$$b = \frac{a'^2(a_1 + 2b_1 + c_1)}{s_1 - s_2},$$

$$c = \frac{a_1 s_1 - 2b_1 - c_1 s_2}{s_1 - s_2},$$

$$d = \frac{\partial C_1}{\partial \xi_0}\sqrt{\frac{-s_2}{s_1 - s_2}} + \frac{\partial C_1}{\partial \eta_0}\sqrt{\frac{s_1}{s_1 - s_2}},$$

$$e = -\frac{\partial C_1}{\partial \xi_0}\sqrt{\frac{s_1}{s_1 - s_2}} + \frac{\partial C_1}{\partial \eta_0}\sqrt{\frac{-s_2}{s_1 - s_2}},$$

$$f = C_1(\xi_0, \eta_0).$$

ξ_0, η_0 being the coordinates of the new origin.

There is one real conic C of the pencil Π whose axes are parallel to the asymptotes of H'. This conic is given by $\lambda = -1/b$, taking H' and C_1 in their reduced forms (2) and (3). Thus

$$C \equiv C' - bH' = 0.$$

Taking C and H' as bases we have

$$\Pi : C + \mu H' \equiv (1 + \mu)(H + \lambda C_1),$$

where

$$\mu = \frac{-1/b + \lambda}{\lambda}.$$

Thus we have finally

$$\Pi : a\xi^2 + 2\mu\xi\eta + c\eta^2 + 2d\xi + 2e\eta + f + \mu h = 0,$$

where

$$h = \frac{H(\xi_0, \eta_0)}{\sqrt{F'}}.$$

The values of the various parameters giving our original conics Γ and $\overline{\Gamma}$ referred to the different bases and the different configurations of reference are assembled as follows:

(I) $\Pi_1 : C_1 + \kappa C_2$; $\quad \Gamma, \overline{\Gamma} : C_1 + \kappa'' C_2 = 0, \quad \kappa'' = \pm i,$

(II) $\Pi_2 : H + \lambda C_1$; $\quad \Gamma, \overline{\Gamma} : H + \lambda'' C_1 = 0,$

$$\lambda'' = \frac{\kappa' \mp i}{\pm i} = -1 \mp i\kappa'', \qquad \kappa' = -\frac{a_1 + c_1}{a_2 + c_2}.$$

After change of origin $\Gamma, \overline{\Gamma} : H' - \lambda'' \sqrt{F'} C' = 0$

(III) $\Pi_3 : C + \mu H'$; $\quad \Gamma, \overline{\Gamma} : C + \mu'' H' = 0,$

$$\mu'' = \frac{-1/b + \lambda'' \sqrt{F'}}{-\lambda'' \sqrt{F'}} = -1 + \frac{1}{b\lambda'' \sqrt{F'}}.$$

We have thus reduced the equation of our conic

(I) $$\alpha\xi^2 + 2\beta\xi\eta + \gamma\eta^2 + 2\delta\xi + 2\epsilon\eta + \zeta = 0$$

containing five effective constants to an equivalent one

(VII) $$a\xi^2 + 2\mu\xi\eta + c\eta^2 + 2d\xi + 2e\eta + f + \mu h = 0$$

containing only two complex constants which are linearly and integrally related.

15. "*Localizing" the Conic in the Complex Pencil.*—We have identified our conic as a member of a complex pencil of conics having two real conics (proper or degenerate) as bases. We now proceed to "localize" it among the double infinity of conics of the pencil. The locus of centers of II is given by

$$\frac{\partial \Pi}{\partial \xi} = a\xi + \mu\eta + d = 0,$$

$$\frac{\partial \Pi}{\partial \eta} = \mu\xi + c\eta + e = 0.$$

Which on the elimination of μ yields

$$C_3 : a\xi^2 - c\eta^2 + d\xi - e\eta = 0,$$

an ellipse or an hyperbola according as C' is an hyperbola or an ellipse.

The double infinity of points on C_3 are the centers of the double infinity of conics of II. The single infinity of real points on C_3 are the centers of the single infinity of real conics of II. The double infinity of imaginary conics have their centers pictured by double infinity of point-pairs $z \rightarrow w$ associated by a Schwarzian reflection over the real branch of C_3.

Thus at this stage of the investigation we are able to "localize" our conic to the extent of determining the picture of the center.

In general: *The point pairs $z_0 \rightarrow w_0$, $w_0 \rightarrow z_0$ picturing the center of an imaginary conic and its conjugate are symmetric with respect to a real central conic, namely the locus of centers of the real conics of a pencil determined by the real and pure imaginary component parts of the conic.*

Such localization of the foci is not so simple since they are known to lie on a bicircular sextic.

CHAPTER IV.

THE REAL REPRESENTATION OF THE COMPLEX CONIC.

In this chapter we consider the reverse conformal transformation $z \rightarrow w$ of the form § 13, II which pictures the ∞^2 points whose coordinates satisfy an equation of the type § 13, I or § 14, VII. Or, stated more explicitly, given a point (ξ', η') satisfying an equation of the above mentioned type, there is simultaneously given, by virtue of the relations $\xi' + i\eta' = w'$ and $\bar{\xi}' + i\bar{\eta}' = z'$, a pair of associated points of the two picture planes (w) and (z). This point-pair $z' \rightarrow w'$ we call the *real picture* of the point (ξ', η') and the ensemble of such pairs picturing the double infinity of points on the conic we call the Real Representation or Real Picture of the complex conic. We shall find that these associated point-pairs may be grouped in their respective planes on two orthogonal families of curves. The two orthogonal nets thus formed in the two picture planes are more or less similar depending on the complexity of the case, being in the case of real conics identical.

We might take for the equation of our conic § 13, I and seek its real representation through the corresponding transformation § 13, II. The method here developed would be found sufficient. But we shall find it somewhat simpler and more interesting to take the so-called canonical form § 13, VII—simpler because we have only one complex quantity among the coefficients, and with the vanishing of its pure imaginary part we have at once the real case—interesting because we shall be able at various points of the development to observe just how the entering of the imaginary affects the configuration.

Accordingly we take for the equation of our conic:

$$(1)\quad a\xi^2 + 2\beta\xi\eta + c\eta^2 + 2d\xi + 2e\eta + \zeta = 0, \qquad \zeta = f + \beta h,$$

writing $\beta = b_1 + ib_2$ in the place of μ in § 13, VII for uniformity of notation. The corresponding equation in z and w given by the relation

$$(2) \qquad w = \xi + i\eta, \qquad \bar{z} = \xi - i\eta;$$
$$\text{or} \quad \xi = \frac{1}{2}(\bar{z} + w), \qquad \eta = \frac{i}{2}(\bar{z} - w),$$

is

$$(1') \qquad \alpha w^2 + 2bw\bar{z} + \gamma\bar{z}^2 + 2\delta w + 2\epsilon\bar{z} + \zeta' = 0,$$

where

$$(3) \qquad \begin{array}{lll} \alpha = a - c - 2i\beta, & \delta = 2(d - ie), & b = a + c, \\ \gamma = a - c + 2i\beta, & \epsilon = 2(d + ie) = \bar{\delta}, & \zeta' = 4\zeta. \end{array}$$

We shall use the following notation for the discriminants of (1) and (1′) and the complementary minors of their elements; the symbols in the first column in each table being for the complex case, $b_2 \neq 0$, and those in the second column for the real case, $b_2 = 0$:

$$(1) \qquad a\xi^2 + 2\beta\xi\eta + c\eta^2 + 2d\xi + 2e\eta + \zeta = 0;$$

$$(1') \qquad \alpha w^2 + 2bw\bar{z} + \gamma\bar{z}^2 + 2\delta w + 2\epsilon\bar{z} + \zeta' = 0,$$

$$\begin{vmatrix} a & \beta & d \\ \beta & c & e \\ d & e & \zeta \end{vmatrix} \equiv \Theta,\ H \cdots \qquad \begin{vmatrix} \alpha & b & \delta \\ b & \gamma & \epsilon \\ \delta & \epsilon & \zeta' \end{vmatrix} \equiv \mathfrak{T},\ T,$$

$$(4) \qquad \begin{array}{l} c\zeta - e^2 \equiv \mathrm{A},\ A \cdots \\ de - \beta\zeta \equiv \mathrm{B},\ B \cdots \\ a\zeta - d^2 \equiv \Gamma,\ C \cdots \\ \beta e - cd \equiv \Delta,\ D \cdots \\ \beta\delta - ae \equiv \mathrm{E},\ E \cdots \\ ac - \beta^2 \equiv \mathrm{Z},\ F \cdots \end{array} \qquad (4') \qquad \begin{array}{l} \gamma\zeta' - \epsilon^2 \equiv \mathfrak{A},\ \mathfrak{A}', \\ \delta\epsilon - b\zeta' \equiv \mathfrak{B},\ \mathfrak{B}', \\ \alpha\zeta' - \delta^2 \equiv \mathfrak{C},\ \bar{\mathfrak{A}}', \\ b\epsilon - \gamma\delta \equiv \mathfrak{D},\ \mathfrak{D}', \\ b\delta - \alpha\epsilon \equiv \mathfrak{E},\ \bar{\mathfrak{D}}', \\ \alpha\gamma - b^2 \equiv \mathfrak{Z},\ F'. \end{array}$$

We have the following relations between the quantities of the two tables.

$$(5) \qquad \begin{array}{lll} \mathfrak{T} = -16\Theta, & \mathfrak{Z} = -4\mathrm{Z}, & \mathfrak{D} = -4[\Delta + i\mathrm{E}], \\ \mathfrak{A} = -4[\mathrm{A} - \Gamma + 2i\mathrm{B}], & \mathfrak{B} = -4[\mathrm{A} + \Gamma], & \\ \mathfrak{C} = -4[\mathrm{A} - \Gamma - 2i\mathrm{B}], & & \mathfrak{E} = -4[\Delta - i\mathrm{E}], \end{array}$$

Relation (2) may be looked upon as a transformation of coordinates, in fact the quantities z and w are called Isotropic Coordinates.

Considering w and z as rectangular coordinates equation (2) expresses an imaginary projective transformation. The determinant of this transformation is

$$J = \begin{vmatrix} 1 & i \\ 1 & -i \end{vmatrix} = -2i.$$

We rewrite (5)

$$\begin{aligned}
&\mathfrak{T} = -J^4\Theta, \quad \mathfrak{Z} = J^2 Z, && \mathfrak{D} = J^2[\Delta + iE],\\
(5')\quad &\mathfrak{A} = J^2[A - \Gamma + 2iB], \quad \mathfrak{B} = J^2[A + \Gamma], &&\\
&\mathfrak{C} = J^2[A - \Gamma - 2iB], && \mathfrak{E} = J^2[\Delta - iE].
\end{aligned}$$

Thus it appears that Θ and Z are invariant under this imaginary projective transformation just as in the real case.

If $Z \neq 0$, we have for the coordinates of the center and its corresponding picture

$$\left.\begin{aligned} \xi_0 &= \frac{\beta e - cd}{ac - \beta^2} = \frac{\Delta}{Z} \\ \eta_0 &= \frac{\beta d - ae}{ac - \beta^2} = \frac{E}{Z} \end{aligned}\right\} \cdots z_0 \rightarrow w_0 \left\{\begin{aligned} w_0 &= \xi_0 + i\eta_0 = \frac{\Delta + iE}{Z} = \frac{\mathfrak{D}}{\mathfrak{Z}}, \\ \bar{z}_0 &= \xi_0 - i\eta_0 = \frac{\Delta - iE}{Z} = \frac{\mathfrak{E}}{\mathfrak{Z}}. \end{aligned}\right.$$

Here again we observe the similarity of notation.

We proceed now to a detailed examination of equation (1′). It is an implicit relation between two complex variables. Solving for w we have

$$w = \frac{1}{\alpha}[-(b\bar{z} + \delta) \pm \sqrt{(b^2 - \alpha\gamma)\bar{z}^2 + 2(b\delta - \alpha\epsilon)\bar{z} + \delta^2 - \alpha\zeta'}],$$

or using the notation of (4) and (4′)

$$\begin{aligned} w &= \frac{1}{\alpha}[-(b\bar{z} + \delta) \pm \sqrt{-\mathfrak{Z}\bar{z}^2 + 2\mathfrak{E}\bar{z} - \mathfrak{C}}],\\ &= \frac{1}{\alpha}[-(b\bar{z} + \delta) \pm 2\sqrt{Z\bar{z}^2 - 2(\Delta - iE)\bar{z} + A - \Gamma - 2iB}], \end{aligned}$$

then

$$w = \frac{1}{\alpha}[-(b\bar{z} + \delta) \pm 2\sqrt{Z}\sqrt{\bar{z}^2 - 2\bar{z}_0\bar{z} + \omega}],$$

$$\omega = \frac{A - \Gamma - 2iB}{Z}, \qquad Z \neq 0,$$

and

$$w = \frac{1}{\alpha}[-(b\bar{z} + \delta) \pm \sqrt{\bar{z} - \bar{z}'}], \qquad \bar{z}' = \frac{\mathfrak{C}}{2\mathfrak{G}}, \qquad Z = 0.$$

If $Z \neq 0$, we have for the roots of the radical

$$\bar{z}', \bar{z}'' = \frac{\mathfrak{G}}{\mathfrak{Z}} \pm \frac{\sqrt{\alpha(-\mathfrak{T})}}{\mathfrak{Z}} = \bar{z}_0 \pm \alpha^{1/2}I, \qquad I = i\sqrt{\mathfrak{T}} \div \mathfrak{Z}.$$

The corresponding values of w are

$$w', w'' = \frac{\mathfrak{D}}{\mathfrak{Z}} \mp \frac{b\sqrt{\alpha(-\mathfrak{T})}}{\alpha\mathfrak{Z}} = w_0 \mp b\alpha^{-1/2}I.$$

If the conic is real, it is known[1] that the points corresponding to the roots of the radical are the two real foci of the conic in the case $Z \neq 0$, and the real finite focus in the case $Z = 0$. In the case of imaginary conics then $z' \rightarrow w'$ and $z'' \rightarrow w''$ are the pictures of the foci, now imaginary, corresponding to the real foci in the real case.

Equation (1′) is of the second degree in either of the variables, hence a two-sheeted Reimann surface is required for complete depiction in case either is expressed as a function of the other. The obvious disadvantage of proceeding in this way is that an irrationality is introduced. We avoid this by introducing a parameter as in the real case. This we effect in the following manner:

The slope of (1) is given by

$$\frac{d\eta}{d\xi} = -\frac{a\xi + \beta\eta + d}{\beta\xi + c\eta + e}.$$

The points where the tangent is perpendicular to the ξ-axis is given by intersection of (1) and $\beta\xi + c\eta + e = 0$. The abscissas of these points are the roots of

(6) $$Z\xi^2 - 2\Delta\xi + A = 0$$

or

(7) $$\xi_1, \xi_2 = \frac{\Delta \pm \sqrt{-c\Theta}}{Z} = \xi_0 \pm ic^{1/2}I,$$

$$I = \sqrt{\Theta} \div Z = \sqrt{-\mathfrak{T}} \div \mathfrak{Z}.$$

The corresponding ordinates are

(8) $$\eta_1, \eta_2 = \eta_0 \mp i\beta c^{-1/2}I.$$

[1] Goursat, Nouvelles Annalles Mathèmatique (1887). Cesaro, Nouvelles Annalles Mathèmatique (1901).

The pictures of these points are

$$(\xi_1, \eta_1), \qquad z_1 \to w_1; \qquad (\xi_2, \eta_2), \qquad z_2 \to w_2;$$

$$\bar{z}_1, \bar{z}_2 = \bar{z}_0 \pm i\frac{c + i\beta}{c^{1/2}} I, \qquad w_1, w_2 = w_0 \pm i\frac{c - i\beta}{c^{1/2}} I.$$

Forming the pencil of lines through (ξ_1, η_1) we have

(9) $$\eta - \eta_1 = \tau(\xi - \xi_1), \qquad \tau = s + it.$$

With the variation of τ through the domain of complex numbers the movable intersection of the line and the conic describes the conic. Thus we introduce the parameter and expressing ξ, η, w and z in terms of it we have

(10) $$\xi = \xi_1 - \frac{2\sqrt{-c\Theta}}{c(a + 2\beta\tau + c\tau^2)}, \qquad w = w_1 - \frac{2\sqrt{-c\Theta}(1+i\tau)}{c(a + 2\beta\tau + c\tau^2)},$$
$$\eta = \eta_1 - \frac{2\sqrt{-c\Theta}\tau}{c(a + 2\beta\tau + c\tau^2)}, \qquad \bar{z} = \bar{z}_1 - \frac{2\sqrt{-c\Theta}(1 - i\tau)}{c(a + 2\beta\tau + c\tau^2)}.$$

Thus we have expressed w and z rationally in terms of a parameter and we proceed at once to a detailed study of the functions thus obtained. We shall find it simpler however to pass through a series of linear transformations of the parameter and thereby transform the functions w and z into forms more frequently met with in function theory. To this end we notice that the common denominator of the above expressions may be written $(c\tau + \beta)^2 + Z$, so our first change of parameter is by the transformation

$$\tau' = c\tau + \beta,$$

and we get

$$\xi = \xi_1 - \frac{2\sqrt{-c\Theta}}{\tau'^2 + Z}, \qquad w = w_1 - \frac{2\sqrt{-c\Theta}(c - i\beta + i\tau')}{c(\tau'^2 + Z)},$$
$$\eta = \eta_1 - \frac{2\sqrt{-c\Theta}(\tau' - \beta)}{c(\tau'^2 + Z)}, \qquad \bar{z} = \bar{z}_1 - \frac{2\sqrt{-c\Theta}(c + i\beta - i\tau')}{c(\tau'^2 + Z)}.$$

Again changing the parameter by the transformation

$$\tau' = i\sqrt{Z}\frac{1 - \tau''}{1 + \tau''}$$

we have after some reductions

$$\xi = \xi_0 + i\frac{c^{1/2}I}{2}\left(\tau'' + \frac{1}{\tau''}\right),$$

$$\eta = \eta_0 - i\frac{c^{-1/2}I}{2}\left[(\beta + i\sqrt{Z})\tau'' + \frac{\beta - i\sqrt{Z}}{\tau''}\right],$$

$$\left|\begin{aligned} w &= w_0 + \frac{I}{2}\left(\tau_1 + \frac{\gamma}{\tau_1}\right), & \tau_1 &= \frac{c - i\beta + \sqrt{Z}}{ic^{1/2}}\tau'', \\ \bar{z} &= \bar{z}_0 + \frac{I}{2}\left(\tau_2 + \frac{\alpha}{\tau_2}\right), & \tau_2 &= \frac{c + i\beta - \sqrt{Z}}{ic^{1/2}}\tau''. \end{aligned}\right.$$

Finally:

$$(11) \qquad \begin{aligned} w &= w_0 + I\sigma_1, & \sigma_1 &= \frac{1}{2}\left(\tau_1 + \frac{\gamma}{\tau_1}\right), \\ z &= z_0 + \bar{I}\bar{\sigma}_2, & \sigma_2 &= \frac{1}{2}\left(\tau_2 + \frac{\alpha}{\tau_2}\right). \end{aligned}$$

For a detailed study of the Riemann surfaces belonging to w and z let us assemble the series of transformations by which we have changed our parameter and examine each turn. We have, first of all, τ the variable slope of the lines of the pencil through (ξ_1, η_1),

$$\eta - \eta_1 = \tau(\xi - \xi_1), \qquad \tau = s + it,$$

whose intersection with the conic gives the ∞^2 complex points of the conic.

The quantities ξ, η, w and z expressed in terms of τ are

$$\begin{aligned} \xi &= \xi_1 - \frac{2\sqrt{-c\Theta}}{c(a + 2\beta\tau + c\tau^2)}, & w &= w_1 - \frac{2\sqrt{-c\Theta}(1 + i\tau)}{c(a + 2\beta\tau + c\tau^2)}, \\ \eta &= \eta_1 - \frac{2\sqrt{-c\Theta}\tau}{c(a + 2\beta\tau + c\tau^2)}, & \bar{z} &= \bar{z}_1 - \frac{2\sqrt{-c\Theta}(1 - i\tau)}{c(a + 2\beta\tau + c\tau^2)}. \end{aligned}$$

Then by a series of transformations,

$$\begin{aligned} \tau' &= c\tau + \beta, \\ \tau' &= i\sqrt{Z}\frac{1 - \tau''}{1 + \tau''}, \\ \tau_1 &= \frac{c - i\beta + \sqrt{Z}}{ic^{1/2}}\tau'', \\ \tau_2 &= \frac{c + i\beta - \sqrt{Z}}{ic^{1/2}}\tau'', \\ \sigma_1 &= \frac{1}{2}\left(\tau_1 + \frac{\gamma}{\tau_1}\right), \\ \sigma_2 &= \frac{1}{2}\left(\tau_2 + \frac{\alpha}{\tau_2}\right), \end{aligned}$$

we bring the functions w and z to the final form

$$w = w_0 + I\sigma_1,$$
$$z = z_0 + \bar{I}\bar{\sigma}_2.$$

We examine these in the reverse order in which they are tabulated. Whatever be the Riemann surfaces belonging to σ_1 and σ_2 equations (11) express merely integral rational transformations of the same. The first consists of an expansion $|I|$ and a rotation $\angle\,(I)$ followed by a translation over the vector w_0. In the second we have first of all a reflection over the axis of reals of σ_2, followed by transformations identical in nature with those of the first equation.

In §§ 4 and 5 we discussed functions very similar to σ_1 and σ_2. There the constant entering, § 4, (7) and § 5, (3) was a real number, c^2. Here we have the complex numbers $\alpha = a - c - 2i\beta$ and $\gamma = a - c + 2i\beta$ appearing in the place of c^2. Let us examine the Riemann surface belonging to σ_1. For the moment let us write $\sigma_1 = u + iv$, $\tau_1 = x + iy$ and $\gamma = a - c + 2i\beta = p + iq$, and resolve the expression into its real and pure imaginary parts

$$u = \frac{1}{2}\left(x + \frac{px - qy}{x^2 + y^2}\right), \qquad v = \frac{1}{2}\left(y + \frac{qx - py}{x^2 + y^2}\right).$$

We seek the orthogonal families of curves in the σ_1-surface corresponding to concentric circles $x^2 + y^2 = r^2$, and the pencil of rays $y = cx$ in the τ_1-plane. For the first we have:

$$(r^4 - 2pr^2 + R^2)u^2 - 4qr^2uv + (r^4 + 2pr^2 + R^2)v^2 - \frac{(r^4 - R^2)^2}{4r^2} = 0,$$

where $R = \sqrt{p^2 + q^2} = |\gamma|$. This is the equation of a conic for which the discriminant, $(r^4 - R^2)^2$ is a quantity which is never less than zero; hence corresponding to the family of concentric circles about the origin in the τ_1-plane is a family of ellipses.

The curves in the σ_1-surface corresponding to the pencil of rays $y = cx$ in the τ_1-plane are given by the equation,

$$2c(cp - q)u^2 + 2q(1 + c^2)uv - 2(p - cq)v^2 - [2cp - q(1 - c^2)]^2/4(1 + c^2) = 0.$$

This equation represents a one-parameter family of hyperbolas for the discriminant, $-(2cp - q + qc^2)^2$, is a quantity which is never greater than zero.

We find further that the real foci of both families of curves is given by $\sqrt{\gamma}$ and furthermore the slope of the principal axis is given by $(R - p)/q$ and this is the tangent of $\angle(\sqrt{\gamma})$. The σ_1-surface then consists of confocal ellipses and hyperbolas corresponding to concentric circles about and radial lines through the origin of the τ_1-plane.

Thus we see that the Riemann surface belonging to the function $\sigma_1 = \frac{1}{2}(\tau_1 + (\gamma/\tau_1))$ is got from the surface belonging to $\frac{1}{2}(\tau_1 + (1/\tau_1))$ by a magnification in the ratio $1 : |\sqrt{\gamma}|$ and a rotation through the angle of $\sqrt{\gamma}$. Similar results hold for the function σ_2 using α instead of γ.

The transformations

$$\tau_1 = \frac{c - i\beta + \sqrt{Z}}{ic^{1/2}}\tau'' \qquad \text{and} \qquad \tau_2 = \frac{c + i\beta - \sqrt{Z}}{ic^{1/2}}\tau''$$

transform the τ_1- and τ_2-planes into themselves, merely interchanging the concentric circles among themselves and the pencil of rays among themselves.

Thus as τ'' describes a circle about the origin in its plane τ_1 and τ_2 describe circles about the origins in their respective planes and in turn σ_1 and σ_2 describe ellipses in their respective planes with centers at the origin; while finally w and z describe ellipses about w_0 and z_0 in their respective planes. Then as τ'' describes a ray of the pencil orthogonal to the concentric circles w and z describe hyperbolas in their respective planes confocal with the ellipses just described.

The Riemann surface belonging to the function

$$\sigma = \frac{1}{2}\left(\tau + \frac{1}{\tau}\right)$$

is two-sheeted, the cross-cut of the surface extending from -1 to $+1$. The unit circle of the τ-plane maps into this line considered as double. Circles outside the unit circle and concentric with it map into ellipses in the upper sheet of the σ-surface. Circles inside the unit circle and concentric with it map into ellipses lying in the lower sheet. So for our functions σ_1 and σ_2.

The Riemann surfaces belonging to them are two-sheeted, the sheets being joined along the lines joining $-\sqrt{\gamma}$ and $+\sqrt{\gamma}$ or $-\sqrt{\alpha}$ and $+\sqrt{\alpha}$ as the case may be. To the circle of radius $|\sqrt{\gamma}|$ of the τ_1-plane corresponds this line joining the branch points $\pm\sqrt{\gamma}$ counted twice. Ellipses in the σ_1-surface, corresponding to circles lying outside the circle of radius $|\sqrt{\gamma}|$ in the τ_1-plane lie in the upper or first sheet of the surface, while those corresponding to circles within the circle of radius $|\sqrt{\gamma}|$ lie in the lower or second sheet. As we have already noted,

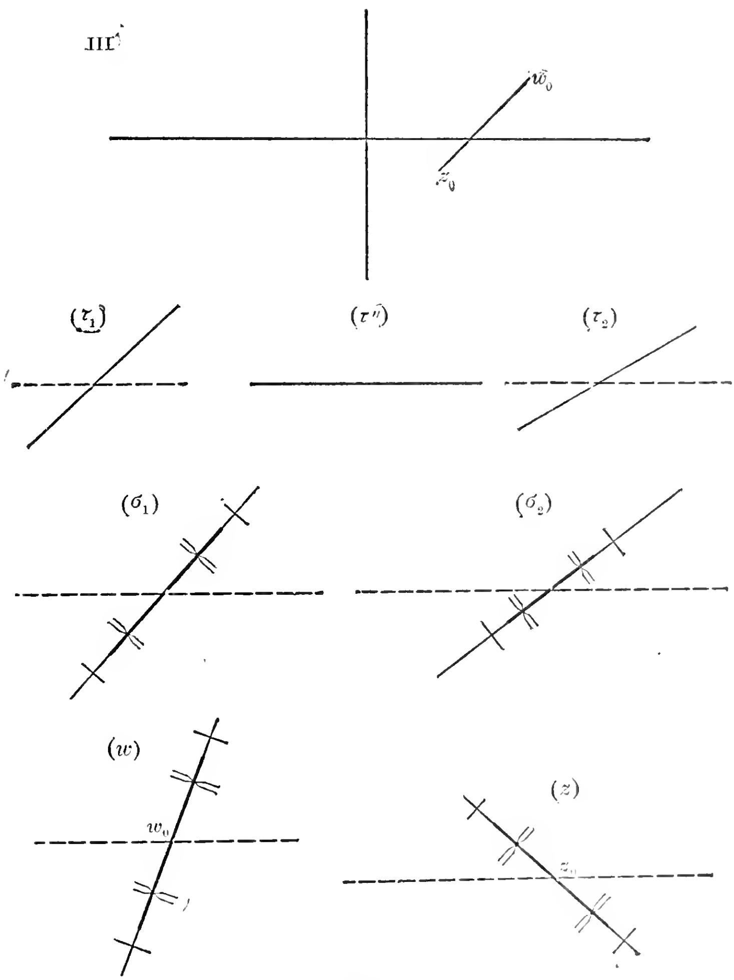

FIG. 3.

the w- and z-surfaces are integral rational transformations of the σ_1- and σ_2-surfaces respectively.

With proper precautions we may consider all these surfaces superposed and with origins common with that of the (ξ, η)-plane. The surfaces discussed are to be considered in no wise organically connected with each other. A schematic diagram of quantities considered is set forth in drawings.

The Case Z = 0.

17. So long as we require that the complex coefficient β in our so-called canonical equation of the conic be a general complex quantity, that is $\beta = b_1 + ib_2$ with b_1 and b_2 different from zero, this case, $Z = 0$ cannot occur; for we must have

$$Z = ac - \beta^2 = ac - b_1^2 + b_2^2 - 2ib_1b_2 = 0,$$

which requires that either $b_1 = 0$ or $b_2 = 0$.

This case requires, then, that either b_1 or b_2 vanish. If $b_2 = 0$, the conic becomes real and we have already considered this case, Chapter I. Again Z may vanish for $b_1 = 0$, that is, for $\beta = ib_2$, pure imaginary. For this case we have

$$Z = ac - \beta^2 = ac + b_2^2 = 0.$$

Hence if $|ac| = b_2^2$ and a and c have opposite signs, Z will be zero. Since $b_2 = \pm i\sqrt{ac}$, the terms of second degree for this case, just as for the real case, form a perfect square:

$$a\xi^2 \pm 2ib_2\xi\eta - c\eta^2 = (\sqrt{a}\xi \pm i\sqrt{c}\eta)^2.$$

Choosing the plus sign we have for the equation of the conic with $Z = 0$:

$$a\xi^2 + 2i\sqrt{ac}\xi\eta - c\eta^2 + 2d\xi + 2e\eta + \zeta = 0.$$

The equation in z and w giving the real representation is

$$a'w^2 + 2b'w\bar{z} + c'\bar{z}^2 + 2\delta w + 2\epsilon\bar{z} + \zeta' = 0,$$

where

$$a' = a + c + 2\sqrt{ac} = (\sqrt{a} + \sqrt{c})^2,$$

$$c' = a + c - 2\sqrt{ac} = (\sqrt{a} - \sqrt{c})^2,$$

$$b' = a - c,$$

$$\delta = 2(d - ie),$$

$$\epsilon = 2(d + ie) = \bar{\delta},$$

$$\zeta' = 4\zeta = 4(f + i\sqrt{ach}).$$

We resort to the use of the parameter in this case as in the preceding, with the same method of introducing it.

The quadratic giving the points where the tangent is vertical reduces for this case to $-2\Delta\xi + \mathrm{A} = 0$, which with

$$\beta\xi - c\eta + e = 0$$

gives for the coordinates of the point

$$\xi_1 = \frac{\mathrm{A}}{2\Delta},$$

$$\eta_1 = \frac{1}{2}\left[\frac{\Gamma}{\mathrm{E}} + \frac{\sqrt{\Theta}}{c^{\frac{1}{2}}\beta}\right].$$

Introducing the parameter as in the preceding section we have

$$\eta - \eta_1 = \tau(\xi - \xi_1), \qquad \tau = s + it,$$

which considered with (1) gives for ξ, η, w and z expressed as functions of τ

$$\xi = \xi_1 + \frac{2\Delta}{(c\tau - \beta)^2} = \xi_1 + \frac{2\sqrt{c\Theta}}{(c\tau - \beta)^2}, \qquad w = w_1 + \frac{2\Delta(1 + i\tau)}{(c\tau - \beta)^2},$$

$$\eta = \eta_1 + \frac{2\Delta\tau}{(c\tau - \beta)^2} = \eta_1 + \frac{2\sqrt{c\Theta}\tau}{(c\tau - \beta)^2}, \qquad z = z_1 + \frac{2\bar{\Delta}(1 + i\bar{\tau})}{(c\bar{\tau} - \bar{\beta})^2}.$$

Making the transformations

$$c\tau - \beta = \tau', \qquad \tau' = \frac{2(c - i\beta)}{\tau_1 - i}, \qquad \bar{\tau}' = \frac{2(c - i\beta)}{\tau_2 - i},$$

of the parameter and applying them respectively to the functions w and z we have after reduction

$$w = w' + \frac{\Delta}{2(c - i\beta)}\sigma_1 = w' + \frac{\sqrt{\Theta}}{2(\sqrt{c} - \sqrt{a})}\sigma_1, \qquad \sigma_1 = \tau_1^2,$$

$$z = z' + \frac{\bar{\Delta}}{2(c - i\beta)}\sigma_2 = z' + \frac{\sqrt{\bar{\Theta}}}{2(\sqrt{c} + \sqrt{a})}\sigma_2, \qquad \sigma_2 = \tau_2^2,$$

where $\Theta = (\sqrt{cd} + i\sqrt{ae})^2$ is the determinant of the conic, $z' \rightarrow w'$ is the picture of the finite focus. The functions σ_1 and σ_2 are well known and have been discussed in Chapter I. The functions w and z are linear integral transformations of the surfaces belonging to σ_1 and σ_2.

18. *The Case*, $b = 0$.—In § 13 we noted that in the case $b = a + c = 0$ our variables w and z were rationally separable. We have

$$\alpha w^2 + 2\delta w = -\gamma \bar{z}^2 - 2\bar{\delta}\bar{z} - 4\zeta,$$

$$\alpha\left(w + \frac{\delta}{\alpha}\right)^2 = \left(\frac{\delta^2}{\alpha^2} + \frac{\bar{\delta}^2}{\gamma^2} - 4\zeta\right) - \gamma\left(\bar{z} + \frac{\bar{\delta}}{\gamma}\right)^2.$$

Let us put

$$w' = \sqrt{\alpha}\left(w + \frac{\delta}{\alpha}\right), \qquad \bar{z}' = \sqrt{\gamma}\left(\bar{z} + \frac{\bar{\delta}}{\gamma}\right),$$

$$\rho^2 = \frac{\delta^2}{\alpha^2} + \frac{\bar{\delta}^2}{\gamma^2} - 4\zeta.$$

Then

$$w'^2 = \rho^2 - \bar{z}'^2.$$

This case is similar to the one discussed by Holzmüller,

$$w = \sqrt{1 - z^2}.$$

The complex quantity ρ however excludes the involutorial property of the case just cited. The simplification of this case is noted by observing equations (9) of section (16).

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