Everymind's Euclid

Euclid's Elements

Books V and VI

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Dedication

For Oliver Byrne,

author of *Doctrine of Proportion*,

whose mistakes clarified my own thinking, and

for Lewis Carroll,

author of *Euclid Book V Proved Algebraically*,

whose refusal to take life too seriously

should be an inspiration to us all.

R. Earle Harris - 2018

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.Instructions

For Learners

Euclid's Books I through IV have given you many useful tools to use anywhere you encounter geometry. But the real power of Euclid comes from Books V and VI.

Analytic geometry appeared when Newton was developing the Calculus. Using these new analytic techniques, mathematicians were finally able to prove things that had eluded their powers in pure geometry. But Newton was able to use pure geometry, especially with the tools of Books V and VI, to prove his theorems for Calculus.

In Book VI, we learn to establish the similarity of figures. This similarity allows us to compare them because their elements are then proportionate. But all of these elements are magnitudes that have nothing to do with number. Book V allows us to compare the proportions of two figures and extensively manipulate these ratios of proportion. And it does this in a way that does not require number.

Book V has been considered the most difficult book of Euclid. This difficulty comes from two things: the desire to preserve Euclid's way of presenting his ideas and the fact that we no longer think the way Euclid did. We can fix this. Book V need not be difficult at all. Proportions are the comparisons of two ratios. In modern notation, this is **exactly** the same as comparing two fractions. And mathematicians such as Augustus De Morgan long ago proved that Euclid's way and the modern way were exactly equivalent in this respect. So we will express Book V in Euclid's notation, prove its theorems in modern notation, and point out where Euclid's concepts are a more powerful superset of our modern fractions. Nothing will be lost and everything gained.

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For Teachers

You must come to your own terms with Book V in order to help others with its ideas. I studied the following, all available on archive.org:

- Euclid for the Use of Schools and Colleges - Todhunter
- The Connexion of Number and Magnitude - De Morgan
- Euclid, Book V, Proved Algebraically - Dodgson (Carroll)
- Doctrine of Proportion or Fifth Book Simplified - Byrne
- Fifth and Sixth Books of Euclid - Hill

My approach is to give the learner everything she needs for understanding and using the rest of Euclid along with everything that will add to her power of expression in all mathematics as it is done today.

For more materials that show the use of Books V and VI, the following are also on archive.org:

- Algebra for the Use of Colleges and Schools - Todhunter
  Especially Chap. 26 on ratio and Chap. 27 on proportion
- Elements of Analytical Geometry - Loomis
  Almost all his proofs use proportions of Book VI
- Modern Geometry - Godfrey and Siddons
  Extensive use of Book VI in what follows from Euclid

The touchstone here is "Can you show Euclid's proportions are more powerful than their simpler form as modern fractions?" This is the imperative.

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Euclid - Book V

If I were a cruel man, I would give you Euclid Book V straight up. Here's Todhunter Proposition 5.1 from Heath's Euclid:

If any number of magnitudes be equimultiples of as many, each to each; whatever multiple any one of them is of its part, the same multiple shall all the first magnitudes be of all the other.

Textbooks of Euclid have tried to be true to Euclid and this is a good thing. But they almost all entirely ignore that we do not think like the Greeks did. Our mathematics is not done the way the Greeks did theirs. We can take their results and use them in our mathematics. But their minds are **alien** to ours.

Almost everything in Euclid Book V can be represented as modern fractions. The danger in using fractions is that Euclid's proportions do far more than our simple fractions. His proportions deal with magnitudes of length, arc, area -- geometric magnitudes to which **no number is attached**. All he needs for his proportions is the knowledge that two figures are similar. In our notation, if A is similar to B we say "A~B". If two figures are similar then their elements are proportional and you can compare them to each other. Let's use first Euclid's notation and then ours in a quick example:

If ΔABC ~ ΔDEF then AB:BC::DE:EF

We say a very similar thing by saying:

If ΔABC ~ ΔDEF then AB/BC = DE/EF

The difference is that Euclid is saying this about magnitudes of lines and in our mathematic, we are saying it about the numbers indicating the length of lines. If ΔABC had sides of length 3,4,5 and ΔDEF had sides three times bigger, Euclid would still say:

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AB:BC::DE:EF

while we would say:

3/5 = 9/15

Then we would reduce the right hand side and wonder what the point was. Here is one of the points. If the sides' lengths were actually irrational lengths that couldn't be exactly expressed in our mathematic's fractions, our fractions would have to be approximations. Euclid's ratios would be exact.

There are even more "points" to learning Euclid's proportions. Because Augustus De Morgan, among others, established that our fractions and Euclid's proportions are completely equivalent under the operations of arithmetic, anything Euclid does with proportions, we can do with fractions. And not just number fractions but algebraic fractions, too. Here again, Euclid is making your consciousness more powerful than you thought it could be. Anywhere you do algebra, you can use the power of Euclid Book V.

As far as I can tell, no Euclid textbook has given problems for Book V. We can fix that. We will use problems from Todhunter's and Bland's algebra texts that show the use of Euclid's ratio and proportion. We are going to pump you up with the power of proportions.

Beyond that, here is our plan for Book V. As in earlier books, I will only include what is actually used in Euclid. But here the propositions will be stated in terms of our mathematics even though I use Euclid's notation. We will then prove his theorems in our equivalent mathematic of fractions. You will keep in mind **always** that Euclid's proportions, even in our fractional form, are more than numeric fractions. I will point out any aspect of this superset of meaning as it comes up. Definitions and axioms will be in both Euclidean and modern notation but will be stated in modern terms -- because I am not a cruel man.

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Axioms

With numbers, these "axioms" are all self-evident. But with magnitudes, Euclid had to make these simple results axiomatic. And without algebra, he had to state everything awkwardly.

a.5.1 Equimultiples of the same or equal magnitudes are equal to each other.

Explanation: Take two magnitudes (a,b) and any number (n), then if a=a (and it always does) $n \times a = n \times a$. And if a=b, then $n \times a = n \times b$

a.5.2 The magnitudes of which the same or equal magnitudes are equimultiples are equal to one another.

Explanation: Same kind of thing. For two magnitudes (a,b) and any number (n) if $na = nb$ then $a=b$

a.5.3 A multiple of a greater magnitude is greater than the same multiple of a lesser magnitude.

Explanation: If $a > b$ then $na > nb$

a.5.4 The magnitude of which a multiple is greater than the same multiple of another is greater than that other magnitude.

Explanation: If $na > nb$ then $a > b$

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Definitions

d.5.1 A lesser magnitude is a **part** or **submultiple** of a greater when the lesser measures the greater exactly.

Explanation: a is a **part** of b if $b = n \times a$ for some $n \in \mathbf{N} = \{ 1, 2, 3, ... \}$

d.5.2 A greater magnitude is a **multiple** of a lesser when the lesser measures the greater exactly.

Explanation: a is a **multiple** of b if $\exists n \in \mathbf{N}: a = n \times b$

d.5.3 Ratio is the comparison of two magnitudes of the same kind: length, area, arc, etc.

Explanation: In Euclid this is a:b and is read "a is to b". In a ratio, the "is to" means "is less than, equal to, or greater than" and Euclid doesn't care which. Whatever the relation is only comes into play when ratios are compared. In modern notation, this is a/b and the "less than, equal to, or greater than" is simply shown by the numbers.

d.5.5 When two ratios, a:b and c:d, have the **same ratio** they are expressed as

$$a:b::c:d$$

Explanation: The full meaning of this is: "a is greater than, equal to, or less than b *in the same way* that c is greater than, equal to, or less than d." This "in the same way" goes further and says that the proportions of the ratios are preserved under all of Book V's operations. So this a:b::c:d says a lot.

In modern terms we can say that if a:b::c:d then

$$a/b = c/d$$

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But the Greeks had more in mind. Because they avoided calculations with incommensurable (irrational) numbers, they could say that if a:b::c:d then there were two whole numbers m and n:

$$ma - nb = 0$$
$$mc - nd = 0$$

In modern mathematics we cannot avoid any numbers. But the above equations are still true for any proportion if m and n are taken not just from the natural numbers (**N**) but from the reals (**R**).

d.5.6 Magnitudes that have the same ratio are called **proportionals** and expressions like a:b::c:d are **proportions**.

Explanation: The whole point of Book V is to show what can be done to proportionals without wrecking the actual geometric proportion of the ratios. In terms of modern fractions, this is: "What can you do to both of them that doesn't destroy their (in)equality?" You can see that, again, fractions are a subset of proportions.

d.5.9 Proportions consist of at least three members.

Explanation: This takes the form a:b::b:c. And this should actually be an axiom.

d.5.10 If a:b::b:c, then a:c is the **duplicate ratio** of a:b.

Explanation: In this case $(a/b)^2 = a/c$

d.5.11 If a:b::b:c::c:d, then a:d is the **triplicate ratio** of a:b and so on. This kind of proportion is called **continued proportion**.

Explanation: In this case $(a/b)^3 = a/d$

d.5.12 Of any number of continued proportionals, the first has to

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the last the **compound ratio** of the series.

Explanation: Think of twenty-five proportional magnitudes

a:b::b:c::c:d::d:e:: ... ::x:y::y:z

then a:z is the compound ratio. Euclid does not say how he arrives at his idea of compound ratio and I've looked for an explanation. But if you consider the series as multiplied fractions:

(a/b)(b/c)(c/d) ... (y/z)

the result is a/z because all the other terms cancel.

This is the "explanation" De Morgan, Carroll, and others give. I believe that Euclid uses this idea of compound ratios, together with duplicate and triplicate ratios, to express area (Book VI) and volume (Book XI) of magnitudes. This is why he doesn't have quadruplicate ratio, as there is no fourth dimension in Euclidean geometry. I will bring up this idea of Euclidean area again when we get to proposition 6.23, which I believe shows his intent for compound ratio.

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Propositions

Proposition 1. Theorem

$\forall$ magnitudes A, B, C, ..., a, b, c, ... and $\forall m \in \mathbf{N}$
If A = ma, B = mb, C = mc, ...
then (A + B + C + ...) = m(a+ b + c + ...)

Proof

(A + B + C + ...) = (ma + mb + mc + ...)   (hyp)
$\therefore$ (A + B + C + ...) = m(a + b + c + ...)   (Distributive Law)

Here, Euclid is proving that the Distributive Law of Arithmetic holds with proportions. Not that he thought of it in those terms. But we do. So the proof is simple. Recall that "$\forall$" means "any, every, or all" and "$\exists$" means "exists."

The main point of these proofs is to explicitly show the idea behind the proposition. Even though I am not using phrases containing the "Let B equal" and "each to each" of direct Greek translation, the proposition in words and symbols is never as clear as the proposition shown and validated by symbols only. Algebra used to be words only. Now algebra is symbols only. You can see why the effort was made to symbolize it.

Proposition 2. Theorem

$\forall$ 6 magnitudes a, b, c, d, e, f and $\forall m,n \in \mathbf{N}$
If a:b::c:d::e:f and a = mb, c = md, e = nb, f = nd
then $\exists r \in \mathbf{N}$: a+e : rb :: c+f : rd

Proof

a:b::c:d::e:f and a = mb, c = md, e = nb, f = nd
$\therefore$ a + e = mb + nb = b(m + n)
c + f = md + nd = d(m + n)
$\therefore$ a+e = (m + n)b c+f = (m + n)d
$\therefore$ a+e : (m + n)b :: c+f : (m + n)d
$\therefore$ r = (m + n)

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Proposition 2. Corollary 1.

∀ magnitudes (A, B, C, ...), (a, b, c, ...), X, x, and ∀ m,n,r,...∈**N**
if A=mX, a=mx, B=nX, b=nx, C=rX, c=rx ...
then ∃ s∈**N**: (A+B+C+...) = sX and (a+b+c+...) = sx

Proof

(A+B+C+...) = mX + nX + rX + ... (hyp)
∴ (A+B+C+...) = X(m + n + r + ...) (5.1)
(a+b+c+...) = mx +nx + rx + ... (hyp)
∴ (a+b+c+...) = x(m +n + r + ...) (5.1)
∴ s = (m+n+r+...)

Proposition 3. Theorem

∀ magnitudes a, b, c, d, A, C and ∀ m,n∈**N**
If a:b::c:d and a=mb, c=md and A=na, C=nc
Then ∃ r∈**N**: A=rb and C=rd

Proof

A = na = n(mb) = (nm)b
C = nc = n(md) = (nm)d
∴ r = nm

Euclid is showing that multiples of magnitudes obey the Associative Law of Arithmetic. Here and in 5.1, he is more or less assuming what he is trying to prove and daring anyone to challenge his arithmetic.

Proposition 4. Theorem

Given proportional magnitudes a:b::c:d and ∀ m,n∈**N**
If magnitudes A=ma, B=nb, C=mc, D=nd then A:B::C:D

Proof

a:b::c:d ∴ a/b = c/d
∴ m/n × a/b = m/n × c/d
∴ ma/nb = mc/nd
∴ A/B = C/D
∴ A:B::C:D

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If Euclid had had algebra, he could have collapsed his twenty-some propositions into a brief algebraic investigation. Without algebra, he has to prove that each tool he wants to use is valid. Keep in mind that throughout these propositions, m or n could be unity, i.e., m/n can be m or 1/n.

Proposition 4. Corollary 1

$\forall$ proportional magnitudes a:b::c:d and $\forall m,n \in \mathbf{N}$

If magnitudes A=ma, C=mc, then A:b::C:d

Proof

a:b::c:d $\therefore$ a/b = c/d

$\therefore$ m × a/b = m × c/d

$\therefore$ ma/b = mc/d

$\therefore$ A/b = C/d

$\therefore$ A:b::C:d

Proposition 4. Corollary 2

$\forall$ proportional magnitudes a:b::c:d and $\forall m,n \in \mathbf{N}$

If magnitudes B=nb, D=nd then a:B::c:D

Proof

a:b::c:d $\therefore$ a/b = c/d

$\therefore$ 1/n × a/b = 1/n × c/d

$\therefore$ a/nb = c/nd

$\therefore$ a/B = c/D

$\therefore$ a:B::c:D

These corollaries are Lewis Carroll's (the mathematician Charles Dodgson). Given the nature of our times, some of you will know that he was supposed to have had his eye on the little girl who was the real-life Alice in Wonderland. Not true. Modern scholarship is divided on whether his attentions were on Alice's nineteen-year-old sister or on Alice's attractive mother. Given that he was told to go away and never come back, the mother is the safer bet as he could have courted the sister in all propriety. And now back to Euclid ...

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Proposition 5. Theorem

$\forall$ 2 magnitudes $a > b$

If $A = ma$ and $B = mb$ then $A - B = m(a - b)$

Proof

$A - B = ma - mb$ (hyp)

$\therefore$ $A - B = m(a - b)$ (5.1)

I have seen two other versions of this proof and none spell out the point any better than this one. The point for Euclid is that the result of (A - B), the equimultiples' difference, is a multiple of the result of (a - b), their parts' difference. Also, a must be greater than b because Euclid cannot draw a negative line or a negative figure in the dirt. His result, of course, remains true for us using any a and b.

Proposition 6. Theorem

$\forall$ 6 magnitudes, a, b, c, d, e, f. $\forall m,n \in \mathbf{N}$ $m>n$

If $a = mc$, $b = md$, $e = nc$, $f = nd$

then either $(a - e) = c$, $(b - f) = d$ or they are equimultiples of c and d

Proof

$a = mc$, $b = md$, $e = nc$, $f = nd$

$\therefore$ $a - e = mc - nc = (m-n)c$

$b - f = md - nd = (m-n)d$

$\therefore$ if $(m-n) = 1$ then $(a - e) = c$ and $(b - f) = d$

else they are equimultiples $(m-n)c$ and $(m-n)d$

I promised you problems for Book V. These next four problems are taken from Isaac Todhunter's *Algebra for the Use of Colleges and Schools,* in Chapter XXVI on Ratio. Think of this as a practical introduction to the use of Euclid's ratio and proportion in algebra. As in the first volume of Everymind's Euclid, the hints, solutions (and diagrams for Book VI) are given in appendices.

The ideas here may be new to you. So don't stare at a problem like a calf at a new gate. See if you can grasp each idea without looking at the hint. But you will need some of the hints. Just go look.

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Problems

1. Write down the duplicate ratio of 2:3 and the subduplicate ratio 100:144.

2. Write down the ratio that is compounded of 3:5 and 7:9.

3. Two numbers are in the ratio of 2:3, and if 9 be added to each of them they are in the ratio of 3:4. Find the numbers.

4. Show that the ratio of a:b is the duplicate ratio of a+c:b+c if $c^2 = ab$.

The lettered propositions in Book V, propositions A - E, are Simson's, who wrote an early and influential Euclid textbook.

Proposition A. Theorem

If a:b::c:d and a >=< b then c >=< d

Proof

a:b::c:d a > b

∴ a/b = c/d

a > b ∴ a/b > b/b ∴ a/b > 1

∴ c/d > 1 ∴ c/d > d/d

∴ dc/d > dd/d ∴ c > d

Sym. a = b and a < b.

This establishes the internal relations of the ratios in a proportion.

Proposition B. Theorem

If a:b::c:d then b:a::d:c

Proof

a:b::c:d

∴ a/b = c/d

∴ b/a = d/c   (1÷ by each)

∴ b:a::d:c

This is proportion taken **inversely** or **by inversion** and extends the use of 5.A.

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Proposition C. Theorem

∀4 magnitudes a, b, c, d.

If a = mb and c = md then a:b::c:d.

Proof

a = mb and c = md

mb:b::md:d (a.5.1,2)

∴ a:b::c:d

The point of Simson's propositions is to create a shorthand for mathematicians working with Euclid. Instead of writing out the lines of this last proof in your proof of some theorem, you can simply write the result and cite (5.C.)

Proposition D. Theorem

If a:b::c:d and a = mb then c = md or if na=b then nc=d.

Proof

a:b::c:d a = mb	a:b::c:d na = b
∴ a/b = c/d	∴ a/b = c/d
∴ mb/b = c/d	∴ a/na = c/d
∴ m = c/d	∴ 1/n = c/d
∴ c = md	∴ nc = d

Let me nag you again about magnitudes. Here a, b could be lines and c, d areas. So if a is a multiple of b in length then c is an equimultiple of d in area. You will need to think in these terms to solve the Book VI problems.

Proposition 7. Theorem

∀2 equal magnitudes, a, b, and any third magnitude x

Because a = b then a:x::b:x

Proof

a = b

∴ a/x = b/x (÷x)

∴ a:x::b:x

Yes, some of these propositions are trivial. You can read the longer versions in traditional Euclids if you wish. But this is all they say.

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Proposition 7. Corollary 1.

∀2 equal magnitudes, a, b, and any third magnitude x

Because a = b then x:a::x:b

Proof

a = b

∴ a:x::b:x (5.7)

∴ x:a::x:b (5.B)

Proposition 8. Theorem

∀ magnitudes, a, b: a > b and any third magnitude c

Because a > b then 1) a:c > b:c and 2) c:b > c:a

Proof

1) a > b

∴ a/c > b/c (÷ c)

∴ a:c > b:c

2) a > b

∴ c/b > c/a (× c/ab)

∴ c:b > c:a

The next four problems are from *Algebraical Problems* by Miles Bland, 1828. Proportions are used in simple algebra problems because if a:b::c:d then the rectangle a•d equals the rectangle b•c which translates, numerically, into ad = bc. In some problems, the text suggests fractions. But often the older notation is easier to work with as the fractions involved are non-intuitive. Problem 8 is like this.

For those of you who have (rightfully) avoided word problems, some advice: In word problems you are always dealing with quantities of something in order to discover what some specific quantity of something specific is. So your equations must result in the required type of quantity. Your unknown should generally be of this type of magnitude. (Probably "always" but I'm a cautious man.)

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You can check this before you even try to do any computation with what you hope is the correct equation. Let's say you need to know how many days and you have quantities of men and days and men/day. You can add days and get days. You can divide men by men/day and get days. But if you multiply men times men/day you get men^2/day, which is an interesting type of magnitude. But it is not, I am sure, the one you are looking for.

Problems

5. A sum of money is to be shared between two persons, A and B, so that as often as A receives 9 pounds, B takes 4. Now it happens that A receives 15 pounds more than B. What are their respective shares?

6. What two numbers are as 2 to 3, to each of which if 4 be added, the sums will be as 5 to 7?

7. The total joint stock of two partners whose particular shares differed by 40 pounds was to the share of the lesser as 14 to 5. Required the shares. (Joint stock means the two put in what they could and when a profit comes in, they split it proportionally to their individual investments.)

8. A person being asked the hour, answered that it was between 5 and 6 and that the hour- and minute-hands were together. What time was it?

Proposition 9. Theorem

$\forall$ 3 magnitudes a, b, c.

If 1) a:c::b:c or 2) c:a::c:b then a = b.

Proof

1) a:c::b:c

$\therefore$ a/c = b/c

$\therefore$ a = b (× c)

2) c:a::c:b

$\therefore$ c/a = c/b

$\therefore$ b = a (× ab/c)

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Proposition 10. Theorem

$\forall$3 magnitudes a,b,c, if a, b have unequal ratios to c: a:c > b:c then a > b and if c:a < c:b then a > b

Proof

a:c > b:c $\therefore$ a/c > b/c $\therefore$ (×c) a > c

c:a < c:b $\therefore$ c/a < c/b $\therefore$ (x ab/c) b < a

Proposition 11. Theorem

Ratios that are equal to the same ratio are equal to each other.

Proof

Follows from Book I, axiom 1. That choice of axiom meant that the Greeks knew equality is transitive, no matter what the equal objects were. They said: If a = c and b = c then a = b. We say: If a = b and b = c then a = c and call this "transitive." Proving this proposition would mean you expect equal things to be unequal.

Proposition 12. Theorem

If any number of magnitudes (a, b, c, ...) are proportional: (a:b::c:d::e:f:: ...) then a : b :: (a+c+e+ ...) : (b+d+f+ ...)

Proof

a:b::c:d::e:f:: ...

$\therefore$ a/b = c/d = e/f = ...

Then each equals some k or a/b = c/d = e/f = ... = k

$\therefore$ a = bk c = dk e = fk ...

$$\therefore \frac{a + c + e + \dots}{b + d + f + \dots} = \frac{bk + dk + fk + \dots}{b + d + f + \dots} = \frac{k(b + d + f + \dots)}{b + d + f + \dots} = k = \frac{a}{b}$$

$\therefore$ a:b::(a+c+e+ ...):(b+d+f+ ...)

Proposition 13. Theorem

$\forall$6 magnitudes a,b,c,d,e,f: if a:b = c:d and c:d > e:f then a:b > e:f

Proof

a:b::c:d $\therefore$ a/b = c/d = k

c:d > e:f $\therefore$ c/d > e/f

c/d > e/f $\therefore$ k > e/f $\therefore$ a/b > e/f

$\therefore$ a:b > e:f

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Proposition 13. Corollary 1.

If a:b > c:d and c:d = e:f then a:b > e:f

Proof

c:d = e:f ∴ c/d = e/f = k

a/b > c/d ∴ a/b > k ∴ a/b > e/f

∴ a:b > e:f

Proposition 14. Theorem

If a:b::c:d and a:b = c:d then as a >=< c so is b >=< d

Proof

Let a > c

a:b = c:d ∴ a/b = c/d

a > c ∴ a/b > c/b (÷b) ∴ c/d > c/b

∴ b > d (× bd/c)

Sym. for a = c and a < c

The next four problems are four more first-degree equation problems from Bland's *Algebraical Problems*.

Problems

9. A Bankrupt owed to two Creditors 140 pounds; the difference of the debts was to the greater debt as 4 to 9. What were the debts?

10. A, B, C make a joint stock; A puts in L.60 (60 pounds sterling) less than B and L.68 more than C; and the sum of the shares of A and B is to the sum of the shares of B and C as 5 to 4.

11. British coins are pounds (L), shillings (s), and pence (d). L.1 = 20s. and 1s. = 12d. When the price of a bushel of barley wanted but 3d. to be to the price of a bushel of oats as 8 to 5, nine bushels of oats were received as an equivalent for four bushels of barley plus 7s. 6d. in money. What was the price of each?

12. Two pieces of cloth of equal goodness, but of different lengths (in yards), were bought, the one for L.5, the other for L.6 10s. Now if the lengths of both pieces were increased by 10, the numbers resulting (lengths) would be in the proportion of 5 to 6. How long was each piece and how much did they cost a yard?

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Proposition 15. Theorem

∀2 magnitudes a,b, ∀n∈**N**, then a:b::na:nb

Proof

a/b = 1 × a/b = n/n × a/b = na/nb ∴ a::b::na:nb

This is really 5.12 where a,b,c,d,e,f,... are a,a,a,... and b,b,b,...
If Euclid had algebra, he wouldn't have to do each possible case in detail. And Euclid's proof of this 5.15 is considerably longer and way more obtuse than this single line that it boils down to. Always be grateful for algebra.

The proportion in the next proposition is in one of Euclid's redundant definition/proposition combos and is called **alternate** or **permuted** ratio. Some books use the Latin **alternando**. In all such cases, the Latin is easy to figure out.

Proposition 16. Theorem

If a:b::c:d then taken **alternately**, a:c::b:d

Proof

a:b::c:d
∴ a/b = c/d ∴ (× b/c) a/c = b/d
∴ a:c::b:d

Proposition 17. Theorem

∀2 magnitudes a,b, if a+b : b :: c + d : d then a:b::c:d

Proof

a+b : b :: c + d : d
∴ (a+b)/b = (c+d)/d ∴ a/b + b/b = c/d + d/d
∴ a/b + 1 = c/d + 1 ∴ a/b = c/d (-1)
∴ a:b::c:d

Proposition 18. Theorem

If a:b::c:d then a+b : b :: c+d : d

Proof

a:b::c:d ∴ a/b = c/d ∴ a/b + 1 = c/d + 1
∴ a/b + b/b = c/d + d/d ∴ (a+b)/b = (c+d)/d
∴ a+b : c :: c+d : d

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Proposition 18 is the converse of Proposition 17. Here again, we can only pity Euclid for not having algebra and being able to reduce Book V to less than ten propositions. At least the next one is not so obvious.

Proposition 19. Theorem

If a:b::c:d then a-c : b-d :: a : b

Proof

a:b::c:d $\therefore$ a/b = c/d = k $\therefore$ a = bk, c = dk

$\therefore$ (a-c)/(b-d) = (bk - dk)/(b-d) = k(b-d)/(b-d) = k

$\therefore$ (a-c)/(b-d) = a/b

$\therefore$ a-c : b-d :: a : b

Proposition 19. Corollary 1

It follows directly that if a:b::c:d then a-c : b-d :: c : d

Proposition E. Theorem

If a:b::c:d then taken by **conversion** a : a-b : c : c-d

Proof

a:b::c:d $\therefore$ a/b = c/d = k $\therefore$ a = bk, c = dk

$\therefore$ a/(a-b) = bk/(bk - b) = b/b × k/(k-1) = d/d × k/(k-1)

= dk/(dk -d) = c/(c-d)

$\therefore$ a : a-b :: c : c-d

I was going to skip this next bit. Other Euclid Book V texts skip it or minimize it. But it turns out to be wildly useful in basic algebra.

Ex Aequali (From Equals)

Euclid has a definition of ex aequali and a definition of ex aequali from equals in proportion and a definition of ex aequali from equals in distorted proportion. And all these say is that if you have two sets of magnitudes in the same proportions, their compounded ratios are equal. And I think we knew this, thanks.

But I was working through an old algebra text and found that ex aequali further shows that proportion is more transitive than you would think. Because it is transitive, you can use it to pass from one set of proportions to another. Let me give you two examples.

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If	A:B::C:D	or	1:3::3:9
and	E:B::F:D	or	2:3::6:9
then	A:E::C:F	or	1:2::3:6

If	A:B::C:D	or	1:3::3:9
and	B:E::D:F	or	3:12:9:36
then	A:E::C:F	or	1:12::3:36

In algebra, these numbers are usually numbers **and** letters and we write them in fractions. You will find that writing them as proportions makes it easier to see the use of ex aequali, as one of the problems below will show. These are simultaneous first-degree equation problems from Bland's *Algebraical Problems*.

Problems

13. Find two numbers, the greater of which shall be to the less as their sum to 42 and their difference to 6.

14. What two numbers are those, whose difference, sum, and product are as the numbers 2, 3, and 5, respectively?

15. A Merchant having mixed a certain number of gallons of brandy and water, found that if he had mixed 6 gallons more of each, he would have put into the mixture 7 gallons of brandy for every 6 of water; but if he had mixed 6 less of each, he would have put in 6 gallons of brandy for every five of water. How many of each did he mix?

16. Find two numbers in the proportion of 5 to 7, to which two other required numbers in the proportion of 3 to 5 be respectively added, the sums shall be in the proportion of 9 to 13; and the difference of those sums = 16.

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Proposition 20. Theorem

∀ magnitudes A,B,C and a,b,c,
if A:B::a:b, B:C::b:c then as A >=< C, a >=< c

Proof

A>C
A:B::a:b B:C::b:c
∴ $A/B = a/b$ $B/C = b/c$
∴ $A/B \times B/C = a/b \times b/c$
∴ $A/C = a/c$
But $A > C$ ∴ (÷C) $A/C > 1$ ∴ $a/c > 1$
∴ (×c) $a > c$
Sym. for $A = C$ and $A < C$

Proposition 21. Theorem

∀ magnitudes A,B,C and a,b,c,
if A:B::b:c, B:C::a:b, then as A >=< C, a >=< c

Proof

A>C
A:B::b:c B:C::a:b
∴ $A/B = b/c$ $B/C = a/b$
∴ $A/B \times B/C = b/c \times a/b$
∴ $A/C = a/c$
But $A > C$ ∴ (÷C) $A/C > 1$ ∴ $a/c > 1$
∴ (×c) $a > c$
Sym. for $A = C$ and $A < C$

This A:B::b:c, B:C::a:b is what Euclid calls **cross order**. Propositions 22 and 23 which follow deal with ex aequali. Proposition 22 shows it as "from equals in proportions" and 23 as "from equals in disordered proportion."

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Proposition 22. Proposition

Given any two sets of magnitudes (A-Z), (a-z),
if A:B::a:b, B:C::b:c, ... Y:Z::y:z, then A:Z::a:z

Proof

A:B::a:b, B:C::b:c, ... Y:Z::y:z
∴ A/B = a/b B/C = b/c ... Y/Z = y/z
∴ A/B × B/C × ... × Y/Z = a/b × b/c × ... × y/z
∴ A/Z = a/z
∴ A:Z::a:z

Proposition 23. Proposition

Given any two sets of magnitudes (A-Z), (a-z),
if A:B::y:z, B:C::x:y, ... Y:Z::a:b, then A:Z::a:z

Proof

A:B::y:z, B:C::x:y, ... Y:Z::a:b
∴ A/B = y/z   B/C = x/y ... Y/Z = a/b
∴ A/B × B/C × ... × Y/Z = y/z × x/y × ... × a/b
∴ A/Z = a/z
∴ A:Z::a:z

More Ex Aequali

Consider this example of Proposition 23:

(2,3,4,2,6,9) (8,12,36,18,24,36)
2:3::24:36
3:4::18:24
4:2::36:18
2:6::12:36
6:9::8:12
∴ 2:9::8:36 or 2/9 = 8/36 = (2×4)/(9×4)

You can see that the relations among proportions need not be simple ones. In the above, four in the first set are paired with equimultiples of 4 and two with equimultiples of 9. But the transitive nature of proportions holds in the result.

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Proposition 24. Theorem

∀6 magnitudes a,b,c,d,e,f,

if a:b::c:d and e:b::f:d then a+e:b::c+f:d

Proof

a:b::c:d e:b::f:d

∴ a/b = c/d e/b = f/d

∴ a/b + e/b = c/d + f/d (1)

∴ (a+e)/b = (c+f)/d

∴ a+e:b::c+f::d

Proposition 24. Corollary 1

Again let a:b::c:d, e:b::f:d then a-e:b::c-f:d

Proof

Sym. from proof of 24 but for (1) substitute

∴ a/b - e/b = c/d - f/d

Proposition 24. Corollary 2

∀2 sets of magnitudes (A,B,C,...) (a,b,c,...) ∀2 magnitudes (X,x):

if A:X::a:x, B:X::b:x, C:X::c:x, ...

then (A+B+C+...):X::(a+b+c+...):x

Proof

A:X::a:x, B:X::b:x, C:X::c:x, ...

∴ A/X = a/x B/X = b/x C/X = c/x ...

∴ A/X + B/X + C/X + ... = a/x + b/x + c/x + ...

∴ (A+B+C+...)/X = (a+b+c+...)/x

∴ (A+B+C+...):X::(a+b+c+...):x

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Proposition 25. Theorem

If a:b::c:d and a is the greatest of (a,b,c,d)

then d is the least of (a,b,c,d) and a+d > b+c

Proof

a:b::c:d and a is the greatest of (a,b,c,d)

$a > b \therefore c > d$ (5.A)

$a > c \therefore b > c$ (5.14)

$\therefore$ d is the least of (a,b,c,d)

a:b::c:d

$\therefore$ a:a-b::c:c-d (5.E)

$a > c \therefore a - b > c - d$ (5.14)

$\therefore$ (+b, +d) $a + d > b + c$

The last four problems are three more with simultaneous first-degree equations and one quadratic. They are all from Bland's *Algebraical Problems.* Number 19 is a monster. But it shows the power of proportions in algebra. And I could have picked a much worse one. But as I said, I am not a cruel man. Problem 20 is a little bit of Euclid Book I for a treat.

Problems

17. A Merchant finds that if he mixes sherry and brandy in quantities which are in the proportion of 2 to 1, he can sell the mixture at 78 shillings a dozen; but if the proportions be 7 to 2, he must sell it at 79 shillings a dozen. Required the price of each liquor.

18. A Corn-factor (and we all know what that is) mixes wheat-flour, which costs him 10 shillings a bushel, with barley-flour, which costs him 4 shillings a bushel, in such proportion as to gain 43 and 3/4 percent, by selling the mixture at 11 shillings per bushel. Required the proportion.

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19. Round two wheels, whose circumferences are as 5 to 3, two ropes are wrapped, whose difference exceeds the differences of the circumferences by 280 yards. Now the larger rope applied to the larger wheel wraps around it a certain number of times, greater by 12 than the number of times the smaller rope wraps around the smaller wheel; and if the larger wheel turns round 3 times as quick as the smaller, the ropes will be discharged at the same time. Required the lengths of the ropes and the circumferences of the wheels.

20. The Captain of a privateer descrying a trading vessel 7 miles ahead, sailed 20 miles in direct pursuit of her, and then observing the trader steering in a direction perpendicular to her former course, changed his own course so as to overtake her without making another tack. On comparing their reckonings it was found, that the privateer had run at the rate of 10 knots in an hour, and the trading vessel at the rate of 8 knots in the same time. Required the distance sailed by the privateer.

Results of Euclid Book V

I hope that you have realized how Euclid's operations on ratios increase your ability and mathematical power when dealing with fractions. Even though all this relates to fractions now, let's recall one last time that Euclid did all of this without numbers and that the values in the ratios were lengths, areas, and volumes. It is also worthwhile to note that although one always begins with ratios of like magnitudes (length A : length B) that this is only the preliminary step. As you manipulate the proportions and relate them to other proportions, you often end up with "length : area :: length : area" and similar.

The following is a brief analysis of what Euclid has established for ratio and proportion, proposition by proposition (remember that A-E were Simson's). Here Euclid is building a foundation for his most powerful book of geometry.

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1. Establishes Distributive Law for addition
2. Extends the Distributive Law for addition
3. Establishes Associative Law of multiplication
4. Produces a proportion from a given proportion
4C1. Ratios still equal if multiplied by a number or its reciprocal
5. Establishes Distributive Law for subtraction
6. Extends Distributive Law for subtraction
7. Extends the syntax of equality
8. Introduces inequality of ratios and their stable relations
9. Converse of 7, further extends syntax of equality
10. Converse of 8, extending syntax of inequality
11. Proves axiom 1 in Book I applies to ratios and proportions
12. Introduces operations on proportions using addition
13. Extends syntax of inequality
14. Establishes external relations of ratios within proportions
15. Establishes reduction of ratios to lowest terms
16. Expands operations between ratios in proportions
17. Establishes subtraction of unity
18. Converse of 17, establishes addition of unity
19. Extends operations on proportions using subtraction
20. Introduces operations between proportions
21. Further extends the concepts of 20
22. Shows transitive nature of proportion between proportions
23. Further extends concepts of 22
24. Establishes addition and subtraction of common denominators
24.C.1 Makes 24 general
25. Extends use of inequalities

Euclid's achievement here is to create a realm of number in a geometry without number. He can now establish proportional relations between geometric figures and operate on those proportions mathematically.

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Euclid - Book VI

Definitions

d.6.1. Similar rectilineal figures are equiangular and have proportional sides about those angles.

d.6.2. Two **reciprocal** triangles or parallelograms (A,B) have sides about their angles such that

side 1 of A : side 1 of B :: side 2 of B : side 2 of A

d.6.3. A line is cut into **extreme and mean ratio** when

whole : greater segment :: greater : lesser

d.6.4. The **altitude** of a figure is the line from its vertex, perpendicular to its base.

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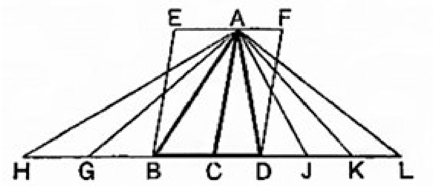
Propositions



Proposition 1. Theorem

Δs or ∥gms of same altitude (alt) have areas in the ratio of their bases.

Proof

ΔABC,ACD ∥gmEC,CF: alt equal

BD(pr): BG = GH = BC DJ = JK = KL = CD Join A[G,H,K,L]

$\therefore$ ΔABC = ΔAGB = ΔAHG (1.38)

$\therefore$ $\forall$n: HC = nBC $\Rightarrow$ ΔAHC = nΔABC

Sym. $\forall$m: CL = mCD $\Rightarrow$ ΔACL = mΔACD

$\therefore$ HC >=< CL $\Rightarrow$ ΔAHC >=< ΔACL

$\therefore$ BC : CD :: ΔABC : ΔACD (d.5.5)

∥gmCE,CF = 2ΔABC,ACD (1.41)

$\therefore$ ∥gmEC : ∥gmCF :: ΔABC : ΔACD

$\therefore$ ∥gmEC : ∥gmCF :: BC : CD (5.11)

Corollary 1.

altΔ = alt∥gm $\therefore$ areaΔ : area∥gm :: baseΔ : base∥gm (1.33,28,6.1)

Note: The arguments of proofs begin to contain a bit more logic.
Line 4: "For any n, if HC = nBC then ΔAHC = nΔABC"

Problems

21. Theorem D

In diagram of 4.10, one Δ is mean proportional of other 2 Δ.

22. Problem

Given: $\forall$$\Delta$ABC

Required: $\forall$O in Δ; ΔOAB = ΔOBC = ΔOCA

23. Theorem *

$\forall$eqSΔABC $\forall$O in Δ OP,OQ,QR$\perp$BC,CA,AB

Then $\sum$O[P,Q,R] constant

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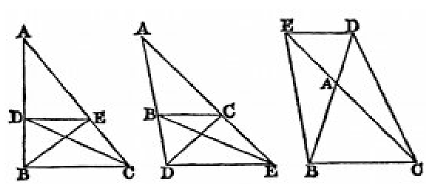
This is the most powerful and the most difficult book of Euclid. The arguments are always in terms of proportions and ratios and they leverage all of the first five books. **Learn how to use Book V**. That's where the power comes from. In some problems they will ask you to show some relation like "parallel" exists, using proportions. Don't beat your brains out if nothing comes to you. Learn from the solution and carry on, keeping calm.

Proposition 2. Theorem

∀ΔABC ∀DE∥BC × AB(pr),AC(pr) @ D,E ∴BD:DA::CE:EA
and conversely.



Proof

1) DE∥BC
 Join BE,CD
∴ΔBDE≡ΔCDE (1.37)
∴ΔBDE : ΔADE :: ΔCDE : ΔADE (5.7)
ΔBDE : ΔADE :: BD : DA (6.1)
Sym. ΔCDE : ΔADE :: CE : EA
∴BD:DA::CE:EA

2) Conversely, BD:DA::CE:EA
∴BD : DA :: ΔBDE : ΔADE   CE : EA :: ΔCDE : ΔADE (6.1)
∴ΔBDE : ΔADE :: ΔCDE : ΔADE (5.11) ∴ΔBDE = ΔCDE (5.9)
ΔBDE,CDE same side DE ∴DE∥BC (1.39)

Problems

24. Theorem

∀ΔABC ∀D∈BC DE,DF∥AB,AC × AB,AC @ E,F
Then ΔBFD : ΔAFE :: ΔAFE : ΔEDC

25. Theorem

∀ΔCAB,DAB ∀E∈AB EF,EG∥AC,AD × BC,BD @ F,G then FG∥CD

26. Theorem

∀ΔABC ∀K∈AB KL,KM∥AC,BC × BC,AC @ L,M KC × LM @ O
Then O on fixed line

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27. Theorem

∀ΔABC BD ×/2∠B × AD⊥BD @ D ED∥BC × AC @ E

Then E mdpt AC

28. Theorem

∀ΔABC ∀DE∥BC × AB,AC @ D,E BE × CD @ F then ΔADF = ΔAEF

29. Theorem

∀ΔABC ∀DE∥BC × AB,AC @ D,E BE × CD @ F

Then AF(pr) ×/2 BC (@H)

30. Theorem

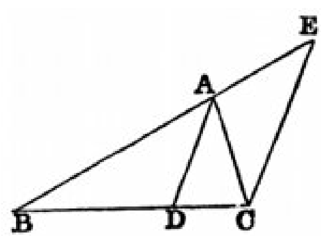
∀4-gonABCD: AB∥DC (AB>DC) EF∥AB × AD,BC @ E,F

Then DE:EA::CF:FB

31. Problem *

Given: ∀ΔABC

Required: P∈AB(pr): PS × AC(pr) @ S BC ×/2 PS



Proposition 3. Theorem

∀ΔABC: AD ×/2∠A × BC @ D ⇒

BD:DC::BA:AC and conversely

Proof

1) AD ×/2∠A

CE∥AD × BA(pr) @ E (1.31)

AC × ∥(AD,EC) ∴ ∠ACE = ∠CAD (1.29)

∠CAD = ∠BAD (hyp) ∴ ∠BAD = ∠ACE

BE × ∥(AD,EC) ∴ ∠BAD = ∠AEC (1.29)

∠BAD = ∠ACE ∴ ∠ACE = ∠AEC ∴ AC=AE (1.6)

ΔBCE: DA∥CE ∴ BD:DC::BA:AE (6.2)

AE = AC ∴ BD:DC::BA:AC (5.7)

2) BD:DC::BA:AC Join AD

BD:DC::BA:AC ∴ BD:DC::BA:AE (6.2)

AD∥EC (con) ∴ BA:AC::BA:AE (5.11) ∴ AC=AE (5.9)

∴ ∠AEC = ∠ACE (1.5)

∠AEC = ∠BAD (1.29) ∠ACE = ∠CAD (1.29)

∴ ∠BAD = ∠CAD ∴ AD ×/2∠A (∠BAC)

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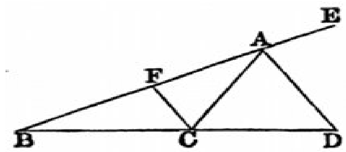
Proposition A. Theorem (Simson's)

∀∆ABC: BA(pr) to ∀E AD ×/2ext∠A(∠CAE) ⇒ BD:DC::BA:AC and conversely



Proof

1) AD ×/2ext∠A
CF∥AD × AB @ F (1.31)
AC × ∥(AD,FC) ∴ ∠ACF = ∠CAD (1.29)
∠CAD = ∠DAE (hyp) ∴ ∠DAE = ∠ACF
FE × ∥(AD,FC) ∴ ∠DAE = ∠AFC (1.29)
∠DAE = ∠ACF ∴ ∠ACF = ∠AFC ∴ AC=AF (1.6)
∆BCF: CF∥AD ∴ BD:DC::BA:AF (6.2)
AF=AC ∴ BD:DC::BA:AC (5.7)

2) BD:DC::BA:AC w/same construction
∴ BD:DC::BA:AF (6.2) ∴ BA:AC::BA:AF (5.11)
∴ AC=AF (5.9) ∴ ∠ACF = ∠AFC (1.5)
∠AFC = ∠DAE (1.29) ∠ACF = ∠CAD (1.29) ∴ ∠CAD = ∠DAE
∴ AD ×/2∠CAE (ext∠A)

Problems

32. Theorem
∀∆ABC D mdpt BC DE,DF ×/2∠ADB,ADC × AB,AC @ E,F
Then EF∥BC

33. Theorem
∀⊙diamAB ∀chordCD⊥AB ∀E∈CD AE(pr),BE(pr) × ⊙ @ F,G
Then 4-gonCFDG: DG:DF::GC:FC

34. Theorem
∀⊙diamAB ∀P∈⊙ Join P[A,B] ∀PC,PD: PA ×/2∠CPD C,D∈AB(pr)
Then AC:BC::AD:BD

35. Problem *
Given: ∀AB Required: ×/3 AB using 6.3

36. Problem *
Given: ∀AB ∀D∈AB
Required: P∈AB(pr): PA:PB::DA:DB

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37. Theorem D

∠BAC = ∠CAD = ∠DAE = ½∟ BE × AC,AD @ C,D: ΔABE ≡ isosΔ

Then BE:BC::BC:CD

38. Theorem

∀ΔABC AD ×/2∠A x BC @ D O mdpt BC

Then OD:OB::AB-AC:AB+AC

39. Theorem *

∀ΔABC AD,AE ×/2∠A,ext∠A × BC(pr) @ D,E O mdpt BC

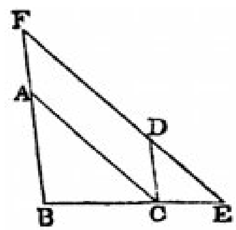
Then OD:OB::OB:OE

40. Theorem D,*

∀ΔABC D,E,F∈BC,CA,AB:

DF,DE;ED,EF;FE,FD make equal ∠ w/BC,CA,AB

Then AD,BE,CF⊥BC,CA,AB



Proposition 4. Theorem

∀ΔABC eq∠ ∀ΔDCE then sides on equal ∠s are proportional

Proof

With Δs as in diagram:

∠BCA = ∠CED (hyp)

∴ ∠ABC + ∠BCA = ∠CED + ∠ABC < 2∟ (1.17)

∴ BA × ED @ F (a.12)

∠ABC = ∠DCE (hyp) ∴ BF∥CD Sym. AC∥FE

∴ FACD ≡ ∥gm ∴ AF=CD AC=FD (1.34)

ΔFBE: AC∥FE ∴ BA:AF::BC:CE (6.2)

AF=CD ∴ BA:CD::BC:CE (5.7) ∴ AB:BC::DC:CE (5.16)

Sym. BC:CA::CE:ED

AB:BC::DC:DE (proven) BA:CA::CE:ED ∴ BA:AC::CD:DE (5.22)

Note: Here Euclid shows sides proportional on apex∠s. He calls the third sides here (BC,CE) **homologous**. All this means is that if two eq∠Δs are oriented in the same way, then the matching sides are homologous. The **important** idea in 6.4 is that if two triangles are equiangular then they are always similar and, by 6.5, conversely. This is **not** true of other figures.

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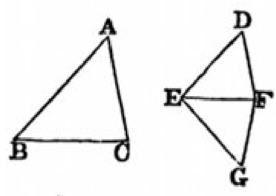
Proposition 5. Theorem

∀ΔABC,DEF: sides proportional as in 6.4⇒ΔABC eq∠ ΔDEF



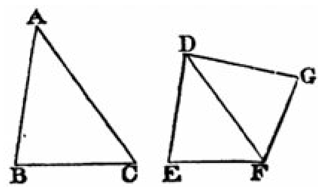
Proof

∠FEG,EFG = ∠ABC,BCA (1.23)
∴ ∠EGF = ∠BAC (1.32)
∴ΔABC eq∠ΔGEF ∴AB:BC::GE:EF (6.4)
AB:BC::DE:EF (hyp) ∴DE:EF::GE:EF (5.11) ∴DE=GE (5.9) Sym. DF=GF
ΔDEF,GEF: DE=GE DF=GF EF=EF ∴ΔDEF eq∠ΔGEF (1.8, 1.4)
ΔGEF eq∠ΔABC (con) ∴ΔDEF eq∠ΔABC

Note: 6.5 is the converse of 6.4

Proposition 6. Theorem

∀ΔABC,DEF: ∠A = ∠D,
BA:AC::ED:DF ⇒ ΔABC eq∠ ΔDEF



Proof

∠FDG,DFG = ∠BAC,ACB (1.23)
∴ ∠G = ∠B (1.32) ∴ΔABC eq∠ ΔDGF ∴BA:AC::GD:DF (6.4)
BA:AC::ED:DF (hyp) ∴ED:DF::GD:DF (5.11) ∴ED=GD (5.9)
∴ΔEDF≡ΔGDF (1.4) ∴ ∠DFG,G = ∠DFE,E
∠DFG = ∠C (con) ∠A = ∠D (hyp) ∴ ∠C,B = ∠DFE,E
∴ΔABC eq∠ ΔDEF

Note: In 6.6, the sides around the equal angles are proportional. In 6.7, the sides around either of the not hypothetically equal angles will be proportional. And in Δ, if two ∠ are equal, so is the third. Recall that figures are similar (~) if they are equiangular (eq∠) and their sides are proportional. And eq∠Δs are always similar. (6.4)

Problems

41. Theorem

∀ΔABC,DEF: baseAB = baseDE AE∥CF
∀KN∥AE × AC,BC,DF,EF @ K,L,M,N then ΔCKL = ΔFMN

42. Theorem

AB∥DC E mdpt DC AC,AE × BE,BD @ F,G then FG∥AB

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43. Theorem

∀AC, ∀fixed A,B,C∈AC, ∀MNC: AM,BN⊥MC
Then BN:AM constant.

44. Theorem

∀A,B ∀MN·|·(A,B) × AB @ C AM,BN⊥MN: AM:BN fixed
Then C fixed.

45. Problem *

Given: ∀A,B,C ratio AF:CG:BH
Required: DE: AF,CG,BH⊥DE in given ratio.

46. Problem *

Given: ∀A,B,C ∀ratio
Required: lineEF: AM,BN⊥EF w/CM:CN in given ratio

47. Theorem D

∀⊙DAE tanDB∥tanEC tanBAC × BD,CE @ B,C BE × DC @ F
Then AF∥DB

48. Theorem

∀4-gonABCD: AB = 2CD AB∥CD then AC ×/3 BD

49. Theorem

∀ΔABC D,E∈AB,AC: BD=CE DE × BC @ F then AB:AC::EF:DF

50. Theorem

∀P,Q ∀AB,CD: AB∥CD ∀PM × AB @ M QN∥PM × CD @ N
Then 1) PM:QN constant and 2) MN on fixed point

51. Theorem

∀⊙C A,B∈⊙C tan@A × tan@B @ T AN⊥CB × CB @ N
Then BT:BC::BN:NA

52. Theorem D

∀ΔABC ∀D∈BC: A,C,D∈⊙P A,B,D∈⊙Q then PA:QA::AC:AB

53. Theorem

APB∥CQD: AP:PB::DQ:QC then PQ,AC,BD concur (meet @ point)

54. Theorem

∀ΔABC AC(pr) to D:CD=AC Join BD MN∥AB × AC,BC @ M,N
MP,NQ∥BD × AB @ P,Q then PA=QB

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55. Theorem

∀rect∟ABDC ∀ΔEAB~ΔAFC: AE,AF homologous (see hint)
EM,FN⊥AB,AC × AB,AC @ M,N EM × FN @ P then P∈AD

56. Theorem D

In diagram 1.43, GE(pr) × HF(pr) ∈ CA(pr)

57. Problem *

Given: ∀ΔABC
Required: O in Δ: ⊥ from O to BC,CA,AB in ratio of X:Y:Z

58. Theorem D

∀⊙A × ⊙B,⊙C @ R,S then RS × BC @ fixed point T

59. Theorem

∀reg. 5-gonABCDE AD × BE @ O then AO::AE::AE::AD

60. Theorem

∀∥gmABCD ∀P,Q∈∀PQ∥AB PA,PB × QB,QC @ R,S then RS∥AD

61. Theorem *

∀ΔABC D mdpt BC ∀P∈AD med∠A BPE,CPF × AC,AB @ E,F
Then EF∥BC

62. Theorem *

∀⊙O,diamAB E mdpt OB ⊙S,diamAE ⊙T,diamEB
common tan PQL × ⊙S,⊙T,AB(pr) @ P,Q,L
Then BL = radius ⊙T

63. Theorem *

∀A,B,CD AC,BD⊥CD AD × BC @ E EF⊥CD × CD @ F
Then ∠AFC = ∠BFD

64. Theorem *

∀∥gmABCD AC × BD @ O AE,CG⊥BD BF,DH⊥AC
Then ∥gmEFGH~∥gmABCD

We are going to move things up a notch. You have the most important tools from Book VI already. The remainder of the problems use only some of the remaining propositions. So we will follow Todhunter's lead and finish up with two big clumps of propositions, each followed by a big clump of problems. Create a synopsis of the results of the propositions and then work on the problems.

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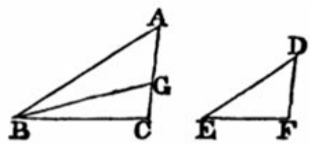
You have done so many proofs and studied so many propositions by now that you won't get that much out of intensely studying the remaining propositions. Learn what they do and how they do it and then go for the problems.

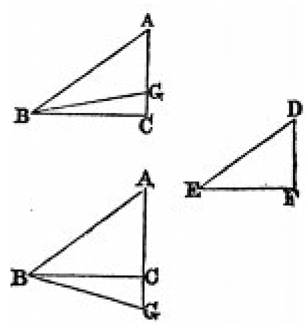
Proposition 7. Theorem



∀ΔABC,DEF: ∠A = ∠D,
AB:BC::DE:EF⇒ΔABC eq∠ΔDEF

Proof

1) ∠C or ∠F < ∟ (1st diagram)
Else ∠ABC≠∠DEF⇒∠ABC > ∠DEF
∠ABG = ∠DEF (1.23)
∠A = ∠D (hyp) ∠ABG = ∠DEF (con) ∴ ∠AGB = ∠DFE (1.32)
∴ΔABG eq∠ΔDEF ∴AB:BG::DE:EF (6.4)
AB:BC::DE:EF (hyp) ∴AB:BC::AB:BG (5.11) ∴BC=BG (5.9)
∴ ∠BCG = ∠BGC (1.5)
∠BCG < ∟ (hyp) ∴ ∠BGC < ∟
∴ ∠AGB > ∟(1.13)
∠AGB = ∠F ∴ ∠F > ∟↴ (hyp)
∴ ∠ABC = ∠DEF ∴ΔABC eq∠ΔDEF



2) ∠C or ∠F ≥ ∠ (same construction)
Sym. w/**1)** BC=BG ∴∠BCG = ∠BGC
∠BCG ≥ ∟(hyp) ∴∠BGC ≥ ∟
∴ΔBCG: ∠G + ∠C > 2∟↴ (1.17)
∴ΔABC eq∠ΔDEF

3) ∠C or ∠F = ∟ (2d diagram)
ΔABC !eq∠ΔDEF⇒∠ABG = ∠DEF (1.23)
Sym. w/**1)** BC=BG ∴∠BCG = ∠BGC (1.5)
∠BCG = ∟ (hyp) ∴∠BGC = ∟
∴ΔBGC: ∠C + ∠G = 2∟↴ (1.17)
∴ΔABC eq∠ ΔDEF

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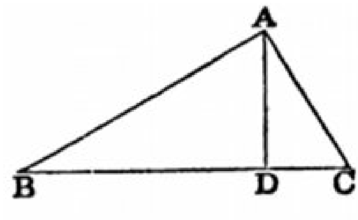
Proposition 8. Theorem

∀ ⊿ABC:∟A AD alt∠A

Then ΔABC~ΔDBA~ΔDAC



Proof

∟BAC = ∟BDA

∠B∈ΔABC,DBA

∴ ∠ACB = ∠DAB (1.32)

∴ΔABC~ΔDAB Sym. ΔABC~ΔDAC

∴ΔABC~ΔDAB~ΔDAC

Corollary 1.

ΔDBA,DAC: BD:DA:DA:DC

ΔABC,DBA: BC:BA:BA:BD

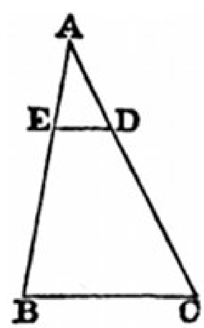
ΔABC,DAC: BC:CA:CA:CD

∴1) alt on ∟ is mean proportional between base segments; and 2) each side is mean proportional between base and adjacent segment of base.

Proposition 9. Problem

Given: ∀AB ∀n∈**N**

Required: AE∈AB: AE = AB/n



Method

∀∠BAC ∀D∈AC

C: AC = n×AD

Join BC DE∥BC

AE required

Proof

ΔABC:BC∥DE

∴CD:DA::BE:EA (6.2)

∴CA:AD::BA:AE (5.18)

CA = nAD

∴BA = nAE (5.D)

∴AE = AB/n

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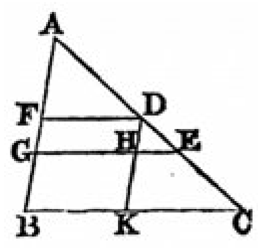
Proposition 10. Problem



Given: ∀AB,AC: AC divided into parts

Required: Divide AB into similar parts

Method

AC divided @ ∀D,E ∀∠BAC

Join BC DF,EG∥BC × AB @ F,G

F,G∈AB required

Proof

DHK∥AB × GE,BC @ H,K (1.31) ∴FH,HB ≡ ∥gm

∴DH=FG HK=GB (1.34)

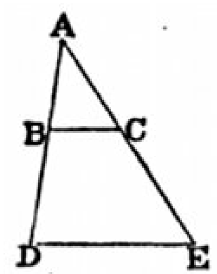
ΔDKC HE∥KC (con) ∴KH:HD::CE:ED (6.2)

DH=FG HK=GB ∴KH:BG::HD:GF ∴BG:GF::CE:ED (5.7)

ΔAGE FD∥GE (con) ∴GF:FA::ED:DA (6.2)

∴BG:GF:CE:ED GF:FA:ED:DA

Proposition 11. Problem



Given: ∀AB,AC: AB < AC

Required: CE: AB:AC::AC:CE

Method/Proof

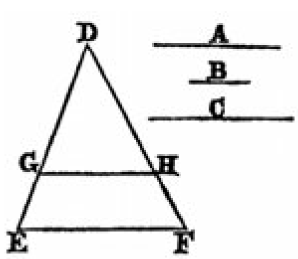
∀∠BAC AB(pr),AC(pr) to D,∀F: BD=AC

DE∥BC × AF @ E CE required

ΔADE BC∥DE (con) ∴AB:BD::AC:CE (6.2)

BD=AC (con) ∴AB:AC::AC:CE (5.7)

Proposition 12. Problem



Given: lines A,B,C

Required: 4th proportional

Method/Proof

∀∠EDF: DG,GE,DH = A,B,C

EF∥GH ∴HF required

ΔDEF: GH∥EF ∴DG:GE::DH:HF (6.2)

DG=A, GE=B, DH=C ∴A:B::C:HF (5.7)

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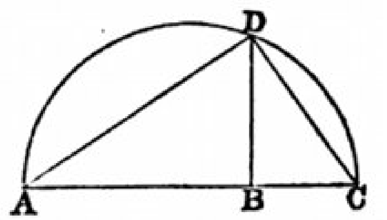
Proposition 13. Problem



Given: ∀AB,BC

Required: BD: AB:BD::BD:BC

Method

AB,BC one line ABC

semi⊙ADC on AC

BD⊥AC × semi⊙ @ D

BD required

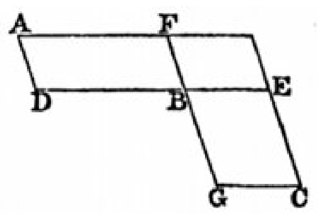
Proof

Join D[A,C] ∴ ∠ADC = ∟ (3.31)

∴ AB:BD:BD:BC (6.8.C1)

Proposition 14. Theorem

Equal ∥gms w/one equal ∠ ⇒sides on equal ∠ proportional and conversely.



Proof

1) equal ∥gms equal ∠s

∥gmAB = ∥gmBC

∠FBD = ∠EBG

DBE colinear ∴ FBG colinear (1.14)

Add ∥FE

∥gmAB = ∥gmBC (hyp) ∴ FE ≡ ∥gm

∴ ∥gmAB : ∥gmFE :: ∥gmBC : ∥gmFE (5.7)

∥gmAB : ∥gmFE :: DB : BE ∥gmBC : ∥gmFE :: GB : BF (6.1)

∴ DB:BE::GB:BF (5.11)

2) ∠FBD = ∠EBG DB:BE::GB:BF w/same construction

DB:BE::GB:BF (hyp)

∴ DB : BE :: ∥gmAB : ∥gmFE GB : BF :: ∥gmBC : ∥gmFE (6.1)

∴ ∥gmAB : ∥gmFE :: ∥gmBC : ∥gmFE (5.11)

∴ ∥gmAB = ∥gmBC (5.9)

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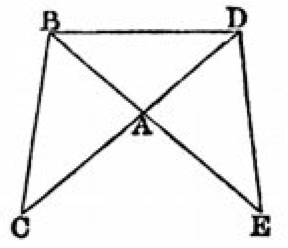
Proposition 15. Theorem



Equal Δs w/one equal ∠⇒sides on equal ∠ proportional and conversely.

1) equal Δs w/one equal ∠

ΔABC = ΔADE ∠BAC = ∠DAE

w/CAD colinear

∴ EAB colinear (1.14) Join BD

ΔABC = ΔADE ∴ΔABC : ΔABD :: ΔADE : ΔABD (5.7)

ΔABC : ΔABD :: CA : AD (6.1)

ΔADE : ΔABD :: EA : AB (6.1)

∴CA:AD::EA:AB (5.11)

2) CA:AD::EA:AB w/same construction

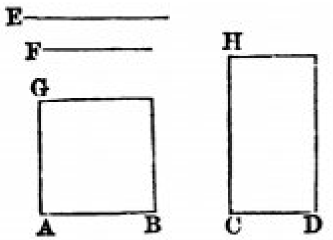
CA:AD::EA:AB (hyp)

CA : AD :: ΔABC : ΔABD (6.1)

EA : AB :: ΔADE : ΔABD (6.1)

∴ΔABC : ΔABD :: ΔADE : ΔABD ∴ΔABC = ΔADE

Proposition 16. Theorem



Lines AB:CD:E:F⇒AB•F = CD•E and conversely

Proof

1) AB:CD:E:F

AG,CH⊥AB,CD: AG=F,CH=E (1.11,3)

Complete ∥gmsBG,DH (1.31)

AB:CD::E:F (hyp) ∴AB:CD::CH:AG (5.7)

∴ ∥gmBG = ∥gmDH (6.14)

∴AB•F = CD•E

2) AB•F = CD•E w/same construction

∴ ∥gmBG = ∥gmDH (con)

∴ ∥gmBG eq∠ ∥DH (con) ∴ ∥gmBG~∥gmDH (6.14)

∴AB:CD::CH:AG ∴AB:CD::E:F (con)

Note: This works because the ∥gms are rect∟s.

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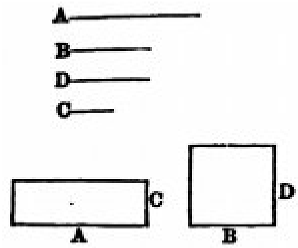
Proposition 17. Theorem

Lines A:B::B:C ⇒ A•C = B^2 and conversely



Proof

1) A:B::B:C

∀D=B ∴ A:B::D:C (5.7)

construction as in 6.16

∴ A•C=B•D (6.16) ∴ A•C=B•B=B^2

2) A•C = B^2 w/same construction

∴ A•C=B•D ∴ A:B::D:C (6.16)

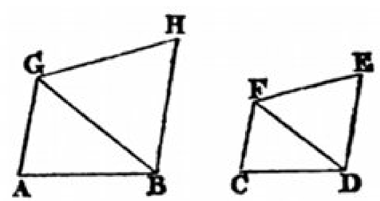
B=D ∴ A:B::B:C

Note: Keep in mind that in Euclid ∀A•B is a **rect∟** with sides equal to A and B. And ∀B^2 is a **square** with sides equal to B. In Euclid, these are **not** numbers; they are **magnitudes**.

Proposition 18. Problem

Given: ∀AB, ∀rectilineal figure

Required: similar figure on AB w/same orientation



1) 4-gonCDEF

Method

Join DF ∠BAG = ∠DCF (1.23)

∠ABG = ∠CDF (1.23)

∴ ΔAGB~ΔCFD (1.32)

∠GBH = ∠FDE (1.23)

∠BGH = ∠DFE (1.23)

∴ ΔBHG~ΔDEF (1.32)

Proof

∠AGB = ∠CFD ∠BGH = ∠DFE (con) ∴ ∠AGH = ∠CFE (a.2)

Sym. ∠ABH,BAG,BHG = ∠CDE,DCF,DEF ∴ ABHG eq∠ CDEF

ΔBAG~ΔDCF (con) ∴ BA:AG::DC:CF (6.4)

Sym. AG:GB::CF:FD BG:GH::DF:FE

∴ AG:GH::CF:FE (5.22) Sym. AB:BH::CD:DE

GH:HB::FE:ED (6.4) ∴ ABHG~CDEF (d.6.1) [cont'd]

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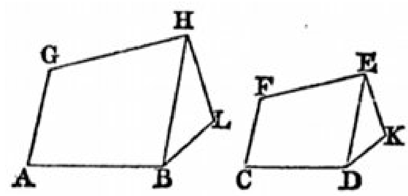
2) 5-gonCDKEF



Method

Join DE

ABHG~CDEF (as above)

∠HBL = ∠EDK (1.23)

∠BHL = ∠DEK (1.23)

∴ ∠L = ∠K (1.32)

Proof

ABHG~CDEF ∠ABH = ∠CDE (d.6.1) ∠HBL = ∠EDK (con)

∴ ∠ABL = ∠CDK (a.2) Sym. ∠GHL = ∠FEK

∴ CDKEF eq∠ ABLHG

ABHG~CDEF (proven) ∴ AB:BH::CD:DE (d.6.1)

BH:BL::DE:DK (6.4) ∴ AB:BL::CD:DK (5.22) Sym. GH:HL::FE:EK

BL:LH::DK:KE (6.4) ∴ ABLHG~CDKEF (6.1)

3) By the same method, for ∀n, a similar n-gon can be constructed.

Problems

65. Theorem

∀⊙O,diamCA x ∀⊙P,diamDE @ B

common tan CD × ⊙O,⊙P @ C,D then DE:DC::DC:CA

66. Problem *

Given: arcEDK∈⊙EDKG, ∀ratio

Required: L∈EDK: chordEL:chordLK = ratio

67. Theorem

∀⊙C ∀A in ⊙C CA(pr) to B: CA:radius⊙C::radius⊙C:CB

Then ∀P∈⊙C ∠CPA = ∠CBP

68. Theorem

∀ΔABC AD alt∠A BD:BA::BA:BC then ∠A = ∟

69. Theorem

∀ΔABC AD alt∠A BD:AD:AD:CD then ∠A = ∟

70. Theorem

∀⊙C,diamAB tanAS∥tanBT: tanSPT × AS,⊙,BT @ S,P,T

Then ∀P SP•PT constant

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71. Problem

Given: ∀ΔABC

Required: FD∥AB × AC,BC @ D,F: FB:FD::FD:FC

72. Theorem D

∀ ⊿ACB ∟C AE,BD⊥AB × BC(pr),AC(pr) @ E,D then ΔECD = ΔACB

73. Theorem

∀ΔABC BE ×/2∠B × AE∥BC,CF∥AB @ E,F then ΔCBE = ΔABF

74. Theorem

∀cyclic4-gonABCD AC × BD @ O

Then ΔAOD~ΔBOC ΔCOD~ΔBOA

75. Theorem *

∀⊙ACBD chordAB × chordCD @ O ∀chordEF∥AB

CE,DF,DE,CF × AB(pr) @ G,H,K,L then OG•OH = OK•OL

76. Problem *

Given: ∀ΔABC

Required: D∈AB: if DE∥BC × AC @ E then ΔADE = ΔBDC

77. Theorem D

∀ ⊿ABC ∟C inscribed squareDFGE: DE∈AB F∈AC

Then DE^2 = AD•BE

78. Theorem

∀ΔABC w/en⊙ AD × BC @ ∀D AE × en⊙ @ E: ∠ACE = ∠ADB

Then AC•AB = AD•AE

79. Theorem

∀∥gmABCD ∀BE × AC,DC,AD(pr) @ F,G,E then EF•FG = BF^2

80. Theorem

∀ΔABC AD ×/2∠C × AB @ D AD(pr) to E: CD•CE = AC•CB

Then if ∠C, AB fixed, E fixed

81. Theorem *

∀cyclic 4-gonABCD CE,DE ×/2∠ACB,ADB × BD,AC @ F,G

Then EF:EG::ED:EC

82. Theorem

∀isosΔABC w/en⊙ AE ×/2∠A × BC,en⊙ @ D,E

Then DA•AE = AB^2

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Proposition 19 is a little tricky to make sense of. Euclid says, "Similar triangles are to one another in the duplicate ratio of their homologous sides." In the proposition, he states the relation of the sides on ∠B,E, making the sides BC,EF homologous by definition. AB,DE are also homologous and so are CA,FD because ΔABC is in the same orientation as ΔDEF. When you orient two figures in the same way, the matching sides are homologous. So you could do this proof from any angle, using the sides about that angle. Duplicate ratio is A:B::B:C. We will have BC:EF::EF::XY where XY will be some third proportion in the duplicate ratio. Then w.r.t area:

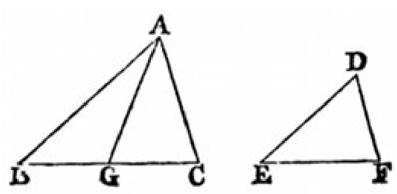
ΔABC : ΔDEF :: BC : XY

Proposition 19. Theorem

∀ΔABC,DEF: ΔABC~ΔDEF ∠B = ∠E AB:BC::DE:EF

∴ BC,EF homologous: BC:EF::EF:XY ⇒ ΔABC : ΔDEF :: BC : XY

Proof



BG: G∈BC BC:EF::EF:BG (6.11)

Join AG AB:BC:DE:EF (hyp)

∴ AB:DE::BC:EF (5.16)

BC:EF::EF:BG (con)

∴ AB:DE::EF:BG (5.11)

∴ ΔABG = ΔDEF (6.15)

BC:EF::EF:BG ΔABC : ΔABG :: BC : BG (6.1) ΔABG = ΔDEF

∴ ΔABC : ΔDEF :: BC : BG

(where BG equals our XY as 3d proportional of duplicate ratio)

Corollary 1

∀AB,CD,EF: AB:CD::CD:EF ⇒ ∀ΔPAB~∀ΔQCD,

ΔPAB : ΔQCD :: AB : EF

Note: Book VI propositions almost all have the form:

geometric relation⇔proportion

They imply each other. So if you have one, you have the other.

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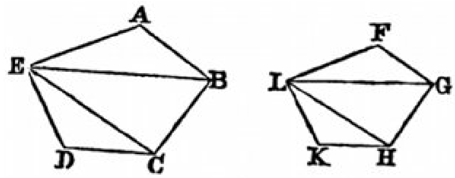
Proposition 20. Theorem

Similar n-gons 1) are divisible into the same similar triangles and 2) are to each other as the duplicate ratio of their homologous sides.



Proof

1) ABCDE~FGHKL

Join BE,EC,GL,LH

ABCDE~FGHKL ∴ ∠BAE = ∠GFL BA:AE::GF:FL (d.6.1)

ΔABE~ΔFGL (6.6 6.4) ∴ ∠ABE = ∠FGL

n-gons similar (hyp) ∴ ∠ABC = ∠FGH (d.6.1) ∴ ∠EBC = ∠LGH (a.3)

ΔABE~ΔFGL ∴ EB:BA::LG:GF

n-gons similar ∴ AB:BC::FG:GH

∴ EB:BC::LG:GH (5.22) ∴ ΔEBC~ΔLGH Sym. ΔECD~ΔLHK

∴ n-gons divisible into same similar Δs

2) ΔABE~ΔFGL ∴ ΔABE : ΔFGL in duplicate ratio of EB:LG (5.19)

ΔEBC~ΔLGH ∴ ΔEBC : ΔLGH in duplicate ratio of EB:LG (5.19)

∴ ΔABE : ΔFGL :: ΔEBC : ΔLGH (5.11)

ΔEBC~ΔLGH ∴ ΔEBC : ΔLGH in duplicate ratio of EC:LH (5.19)

ΔECD~ΔLHK ∴ ΔECD : ΔLHK in duplicate ratio of EC:LH (5.19)

∴ ΔEBC : ΔLGH :: ΔECD : ΔLHK (5.11)

∴ ΔABE : ΔFGL :: ΔEBC : ΔLGH :: ΔECD : ΔLHK (5.11)

∴ ΔABE : ΔFGL :: ABCDE : FGHKL (5.12)

∴ ABCDE : FGHKL in duplicate ratio of AB:FG

Note: Or in duplicate ratio of any two homologous sides of either the n-gons or of two of their matching triangles as all are in the same relation. More importantly, this proposition allows Euclid to relate the areas of any similar n-gons.

Corollary 1.

Sym. by the same method, the same is true of n-gons for ∀n∈**N**.

Corollary 2.

∀AB:CD::CD:EF then AB : EF :: n-gon on AB : similar n-gon on CD

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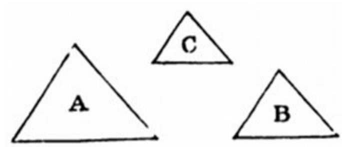
Proposition 21. Theorem

∀n-gons A,B,C n ≥ 3: A~C B~C ⇒ A~B

Proof



A~C ∴ A eq∠C and sides on equal angles proportional

B~C ∴ B eq∠C and sides on equal angles proportional

∴ A eq∠ B (a.1) and sides on equal angles proportional (5.11)

∴ A~B (d.6.1)

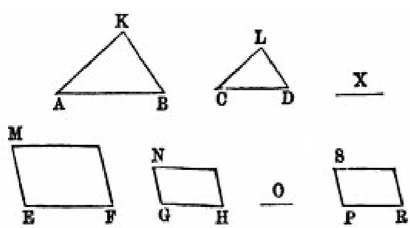
Proposition 22. Theorem

AB:CD::EF:GH: similar figures on AB,CD, any other similar figures on EF:GH ⇒ figAB : figCD :: figEF : figGH and conversely

Proof

1) AB:CD::EF:GH KAB~LCD MF~NH

X,O: AB:CD::CD:X EF:GH::GH:O (con)



∴ CD:X::GH:O (5.11)

AB:CD::EF:GH (hyp)

∴ AB:X::EF:O (5.22)

AB:X::KAB:LCD (6.20.C2)

EF:O::MF:NH (6.20.C2)

∴ KAB:LCD::MF:NH (5.11)

2) KAB:LCD::MF:NH

PR: AB:CD::EF:PR SR~NH (6.18)

AB:CD::EF:PR KAB~LCD MF~SR

∴ KAB:LCD::MF:SR (from **1**)

KAB:LCD::MF:NH (hyp) ∴ MF:SR::MF:NH (5.11) ∴ SR = NH (5.9)

SR~NH ∴ PR = GH

AB:CD::EF:PR PR=GH ∴ AB:CD::EF:GH

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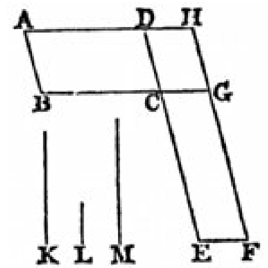
Proposition 23. Theorem

∀ ∥gmAC,CF: AC eq∠ CF⇒AC:CF = compound ratio of their sides

Proof



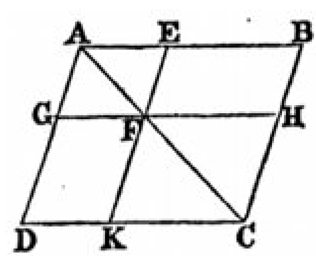
∥gmADCB,CGEF: ∠BCD = ∠ECG
BC,CG one line ∴ DC,CE one line (1.14)
Complete ∥gm DG
∀K,L,M: K:L::BC:CG L:M::DC:CE (6.22)
∴ K:L::L:M = BC:CG::DC:CE
∴ K:L::L:M = ratios of sides
∴ K:M = compound ratio of ∥gm sides
∥gmAC : ∥gmCH ::BC:CG (6.1) :: K:L (con) ∴ ∥gmAC:∥gmCH::K:L
∥gmCH : ∥gmCF :: DC:CE (6.1) :: L:M (con) ∴ ∥gmCH:∥gmCF::L:M
∴ ∥gmAC : ∥gmCF :: K : M (5.22) = compound ratio of their sides

Note: Here again, Euclid uses A:B::B:C for area. In our terms we have areaAC/areaCF = K/M where these are numbers. K/M is some r∈**R**. ∴ areaAC = r×areaCF. But Euclid has no numbers. So his result is only for comparing areas.

Proposition 24. Theorem

∀ ∥gms on diagonal of ∀ ∥gm are similar to e.o and to the whole ∥gm

Proof



∥gmEG,HK∈AC of ∥gmABCD
DC∥GF ∴ ∠ADC = ∠AGF (1.29)
BC∥EF ∴ ∠ABC = ∠AEF (1.29)
∠BCD,EFG = ∠BAD (1.34)
∴ ∥gmABCD eq∠ ∥gmAEFG
ΔBAC,EAF: ∠BAC = ∠BAC ∠ABC = ∠AEF ∴ΔBAC eq∠ ΔEAF (1.32)
∴ AB:BC::AE:EF (6.4)
opp sides ∥gm equal (1.34)
∴ AB:AD::AE:AG DC:CB::GF:FE CD:DA::FG:GA (5.7)
∴ ABCD~AEFG (d.6.1) Sym. ABCD~FHCK ∴ AEFG~FHCK (6.21)

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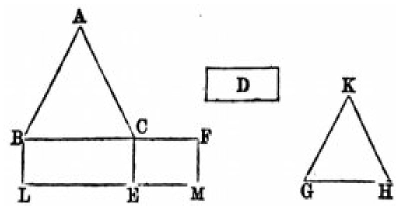
Proposition 25. Problem

Given: two rectilineal figures ABC,D

Required: figure KGH: KGH~ABC KGH = D

Method



∥gmBCEL = ABC

∥gmCEMF = D:

∠FCE = ∠CBL (1.45.C1)

∴ BC,CF;LE,EM colinear

GH: BC:GH::GH:CF (6.13)

KGH~ABC (6.18) required

Proof

BC:GH::GH:CF (con) ∴ ∥gmBC : ∥gmCF :: ABC : KGH (6.20.C2)

BC : CF :: ∥gmBE : ∥gmCM (6.1) ∴ ABC:KGH::BE:CM (5.11)

ABC = CE ∴ KGH = CM = D (con)

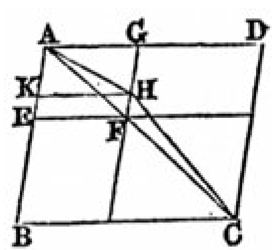
∴ KGH~ABC KGH = D

Note: From previous propositions we know this method will work for any two given n-gons.

Proposition 26. Theorem

∀2 ∥gms w/common∠ and same orientation

⇒∥gms∈same diagonal



Proof

∥gmABCD,AEFG common∠A

∴ AF∈AC

Else ∥gmBD diagonalAHC ≠ AFC

GF × AHC @ H HK∥AD x AB @ K

∴ diagonalAHC∈ABCD,AKHG

∴ ABCD~AKHG (6.24) ∴ DA:AB::GA:AK

ABCD~AEFG (hyp) ∴ DA:AB:GA:AE ∴ GA:AK::GA:AE (5.11)

∴ AE=AK⮧ (greater = lesser) ∴ AF∈AC

Euclids written in our western civilization omit propositions 6.27, 6.28, and 6.29. If you look them up, you will see why. They are an example of how the problems of one culture may have no meaning for another culture. It could be that these were very important,

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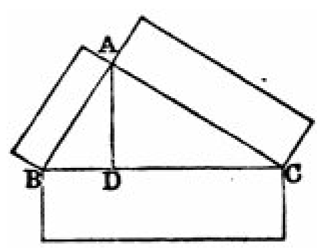
perhaps for temple construction. But our minds are unable to make meaningful sense of these Greek relations. That does not make them or us stupider. We are simply different in our thinking.

Proposition 30. Problem

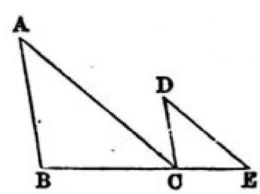
Given: ∀AB
Required: C: AB:AC::AC:CB
Method/Proof
C∈AB: AB:BC = AC^2 (2.11)
∴ AB:AC::AC:CB (6.17)
Note: Euclid calls this the **extreme and mean ratio**.

Proposition 31. Theorem



∀ ⊿ABC ∟A w/similar figures on AB,BC,CA⇒figBC = figAB + figAC
Proof
AD alt∠A (1.12) ∴ ⊿ABD,CAD~⊿CBA
⊿CBA~⊿ABD ∴ CB:BA::BA:BD (d.6.1)
∴ CB : BD :: figBC : figAB (6.20.C2)
∴ BD : BC :: figAB : figBC (5.B)
Sym. CD : BC :: figAC : figBC
∴ BD+DC : BC :: figAB + figAC : figBC (5.24)
∴ BC : BC :: figAB + figAC : figBC ∴ figAB + figAC = figBC (5.A)

Proposition 32. Theorem



∀ΔABC,DCE: BA:AC::CD:DE AB,AC‖DC,DE
∠A = ∠D ⇒BC,CE colinear
Proof
AC × ‖(AB,DC) ∴ ∠BAC = ∠ACD (1.29)
Sym. ∠ACD = ∠CDE ∴ ∠BAC = ∠CDE
∠A = ∠D BA:AC::CD:DE (hyp) ∴ΔABC~ΔDCE (6.6)∴ ∠ABC = ∠DCE
∠BAC = ∠ACD ∴ ∠ACE = ∠ABC + ∠BAC
∴ ∠ACB + ∠ACE = ∠ABC + ∠BAC + ∠ACB
∴ ∠ACB + ∠ACE = 2∟ (1.32) ∴ BCE colinear

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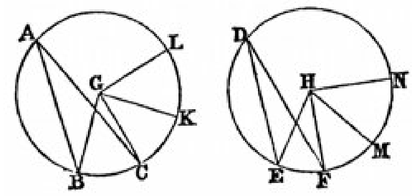
Proposition 33. Theorem

∀equal ⊙s 1) ∠s on⊙, ∠s on center, and 2) sectors in same ratio as their arcs.



Proof

1) ⊙G = ⊙H

arcBC = arcCK = arcKL arcEF = arcFM = arcMN

∴ ∠BGC = ∠CGK = ∠KGL (3.27)

∴ ∀n: BL = n×BC⇒∠BGL = n×∠BGC

Sym. ∀m :EN = m×EF⇒∠EHN = m×∠EHF

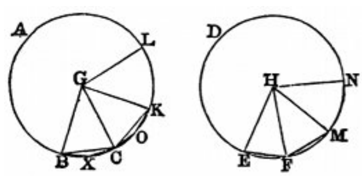
∴arcBL >=< arcEN⇒∠BGL >=< ∠EHN (3.27)

∴arcBC : arcEF :: ∠BGC : ∠EHF (d.5.5)

∠BGC : ∠EHF :: ∠BAC : ∠EDF (5.15, 3.20)

∴arcBC : arcEF :: ∠BGC : ∠EHF :: ∠BAC : ∠EDF

2) ∀X,O∈arcBC,arcCK



ΔBGC,CGK: BG,GC = CG,CK

∠BGC = ∠CGK ∴BC=CK (1.4)

arcBC = arcCK (con)

∴ ∠BXC = ∠COK (3.27)

∴segmentBXC~segmentCOK (d.3.11)

BC=CK ∴segmentBXC~segmentCOK (3.24, d.3.11)

ΔBGC = ΔCGK (proven)

∴sectorBGC = sectorCGK (a.2) = sectorKGL (sym.)

Sym. sectorEHF = sectorFHM = sectorMHN

∴ ∀n: arcBL = n×arcBC⇒sectorBGL = n×sectorBGC

Sym. ∀m: arcEN = m×arcEF⇒sectorEHN = m×sectorEHF

∴arcBL >=< arcEN⇒sectorBGL >=< sectorEHN

∴arcBC : arcEF :: sectorBGC : sectorEHF (d.5.5)

Note: Recall that any statement "X⇒Y" is read "if X then Y." And for "∀n: BL = n×BC", we read "if for any n, BL is the multiple n of BC."

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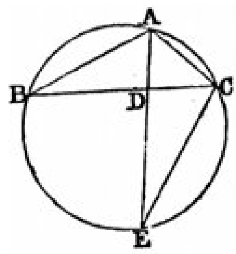
Proposition B. Theorem (Simson's)

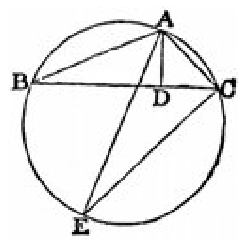


∀ΔABC AD ×/2∠A × BC @ D⇒
AB•AC = BD•BC + AD^2

Proof

en⊙ACB (4.5) AD(pr) to E∈en⊙ Join EC
∠BAD = ∠EAC (hyp) ∠ABD = ∠AEC (3.21)
∴ΔBAD~ΔEAC (1.32) ∴BA:AD::EA:AC (6.4)
∴BA•AC = AD•EA (6.16) = ED•DA + AD^2 (2.3)
ED•DA = BD•DC (3.35) ∴AB•AC = BD•DC + AD^2

Proposition C. Theorem (Simson's)

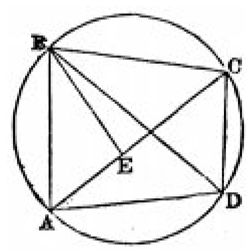


∀ΔABC w/en⊙ AD alt∠A × BC @ D⇒
AB•AC = AD•diam⊙ABC

Proof

diamAE Join EC
∠BDA = ∠ECA (3.31) ∠ABD = ∠AEC (3.21)
∴ΔABD~ΔAEC ∴BA:AD::EA:AC (6.4)
∴BA•AC = AD•EA (6.16)

Proposition D. Theorem (Simson's)



∀cyclic 4-gonABCD Join AC,BD⇒
AC•BD = AB•CD + AD•BC

Proof

∠ABE = ∠DBC (1.23)
∴ ∠EBD + ∠ABE = ∠DBC + ∠EBD
∴ ∠ABD = ∠EBC (a.2)
∠BDA = ∠BCE (3.21) ∴ΔABD~ΔEBC
∴AD:DB::EC:CB (6.4) ∴AD•CB = DB•EC (6.16)
∠ABE = ∠DBC (con) ∠BAE = ∠BDC (3.21) ∴ΔABE~ΔDBC
∴BA:AE::BD:DC (6.4) ∴BA•DC = AE•BD (6.16)
∴AD•CB + BA•DC = DB•EC + AE•BD = BD•AC

Note: AE + EC = AC

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Problems

83. Theorem D,*

In diagram of 4.10, CF∥BD × AD @ F (∴ FB ×/2∠ABD)

Then ΔACF : BCFD :: BD : BA

84. Theorem

In diagram of 6.24, EG,HK × AF,CF @ P,O then EG∥KH

85. Problem

Given: ∀ΔABC

Required: ED⊥AB: ED ×/2ΔABC

86. Problem

Given: ∀⊙A ∀B in ⊙A ∀ratio

Required: chordCBD: CB:BD = given ratio

87. Problem

Given: ∀⊙A ∀B outside ⊙A

Required: secantBCD: C,D∈⊙A BC=CD

88. Problem

Given: baseBC ∠A rect∟AC•AB

Required: implied Δ (without using 1.3)

89. Theorem

∀eqSΔABC w/en⊙ ∀P∈en⊙: P·|·(A,C) then PB = PA + PC

90. Theorem

∀isosΔABC BD,CD⊥AB,AC then BC•AD = 2(AB•DB)

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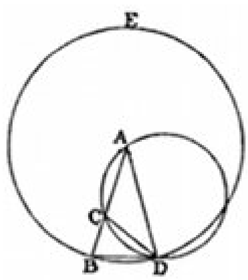
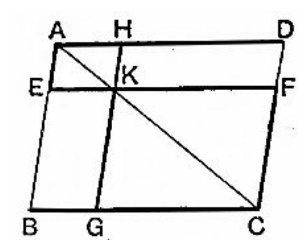
Problem Diagrams

For Euclid Book VI

21. 	**37.** Vertical AB. Then 3 lines, each separated by 45°, off of A. Join BE to make isosΔABE. Label C,D.
40. Draw ΔABC 3-4 lines tall. Then AD,etc.⊥BC,etc. Then join ΔDEF. This gives you the equal angles. Now prove it.	**47.** Circle with 2 ‖ tangents. Draw third tangent × ‖s. Label A-E. BE×DC @ F. Join AF.
52. ⊙P×⊙Q@A,D. Line BDC. Join A[B,C]	**56.** 
58. Make ⊙C much smaller than ⊙B to keep T on page.	**72.** Make ΔACB two lines tall and almost an isosΔ to keep D,E on the page.
77. Draw the square first.	**83.** Same as #21 above.

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Problem Hints

1. Duplicate ratio is composed of the squares of the values. This comes from definition 5.10:

$$a:b::b:c \therefore a/b = b/c \therefore ac = b^2 \text{ or } (a/b)^2 = a/c$$

Subduplicate is the square root. Sym. triplicate, subtriplicate, etc. You should be able to show from a:b::b:c::c:d that $(a/b)^3 = a/d$.

2. Compound ratio is described in its definition in Book V.

3. If numbers are in a ratio of n:m, the first is n × some quantity and the second is m × the same quantity.

4. Calculate the duplicate ratio of (a+c) : (b+c) as an algebraic fraction.

5. 9 is to 4 as what is to what?

6. The numbers being as 2:3 are 2x and 3x.

7. total : lesser :: 14 : 5

8. rate minute hand : rate hour hand :: improper fraction : fraction

9. A + B = 140 and A - B : A :: 4 : 9

10. A+B : B+C :: 5 : 4 and A = B - 60 etc.

11. **b**arley, oa**t**s: **b**+3 : **t** :: 8 : 5 (Calculate in d.)

12. x = cost/yard in 10s. Then the lengths are 10/x and 13/x

13. x:y::x+y:42 etc.

14. x-y : x+y :: 2 : 3 etc.

15. x,y = number of gallons of brandy, water. x+6 : y+6 :: 7 : 6 etc.

16. The four numbers are 5x and 7x, 3y and 5y.

17. x,y = price/dozen sherry, brandy. 2x + 1y = 3 dozen × 78 etc.

18. x:y = wheat:barley. 10x + 4y = cost in s. of x+y bushels, etc.

19. 5x, 3x circumferences in yards. 2x + 280 diff. of ropes.

20. Line ABDC, AB = 7, AD = 20, trader's ⊥ tack CE. Join DE.

21. If A : B :: B : C then B is mean proportion between A and C.

22. Need alt of inner Δ = ⅓altΔABC from all sides.

23. Strive a bit and look up the solution. Learning experience.

24. Use sides as in #21.

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25. Need CF:FB::DG:GB
26. Need 3d line on O ∥ sides. Then proportions.
27. ΔAFC FC∥DE 6.2
28. Need to relate both Δ to ΔBDF
29. ΔAFB = ΔAFC (#28) Need ΔBFH = ΔCFH
30. Add line for Δ and put a ∥ line through it.
31. Need diagram like 6.2 that forces BC to ×/2 a PS.
32. Need AF:FC::AE:EB
33. 6.3 on ΔDGC,DFC
34. Use 6.3, 6.A for two proportions with CP:PD
35. ∀magnitudeM: ⊙A,M ⊙B,2M
36. Solution is a lot like #34.
37. Identify equal lines and ∠s. Then 6.3 and 6.A.
38. 6.3 then use A-B:A+B::C-D:C+D
39. Line 2 in solution of #38.
40. AD×FE@G DE(pr) to ∀H Then 6.A. Need ∠GDB = ∠GDC.
41. 6.4 #30 1.38
42. 2 proportions by 6.4. then ∥ by 6.2.
43. 6.4
44. Variant of #43.
45. New kind of problem. Try a bit then study solution.
46. Like #45 but harder.
47. Need BA:AC::BF:FE for ∥ (6.2). Use eq∠Δs.
48. Similar Δs and AB:CD.
49. Need ∥ line for 6.4.
50. 1) similar Δs 2) PQ×MN@R
51. Need ΔBAN~ΔBCT. Find a cyclic 4-gon on new ⊙.
52. Need ΔAQB~ΔAPC.
53. PQ×AC@L PQ×BD@M Need L ≡ M.
54. CE∥AB×BD@E   Need MP:NQ::PA:QB
55. Homologous: Δs oriented similarly on AE,AF. Similar Δs twice.
56. Method of #53. Need CP:KP::CQ:KQ (lines × @ P,Q)
57. Ratio lesson. Think about it and then study solution.

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58. RS×⊙C@S,P Need AB∥CP.

59. Add en⊙ then show ΔAOE~ΔADE.

60. QR:BR::PQ:AB Need QR:BR::RS:BC for ∥ by 6.2.

61. EM,FN∥AD × BC @ M,N Need EM=FN.

62. TL - TQ = BL in proportion of 5.17.

63. 5.23 Need ΔBDF~ΔACF.

64. Need ΔFEH~ΔBAD. Sym. other half of ∥gm.

65. Need ΔACD~ΔCDE.

66. Lesson in ratios. Try hard for a bit and then study solution.

67. 6.6 radius = CP

68. 6.6 similar Δs

69. similar Δs

70. 6.8.C1 Need ∠SCT = ∟

71. CE∥AB: CE:CB:CB:BA

72. ΔECA eq∠ ΔBCD

73. Method of #72 with an extra step from Book V.

74. Use the cyclic 4-gon's en⊙.

75. EF × CD @ M then think about ΔCEM and ΔCFM.

76. Lesson in proofs. Study 2.11. Assume ∀D∈AB w/the equal Δs.

77. ΔADF~ΔGEB

78. Need Δ ↘ΔBAD.

79. Need two pairs of similar triangles.

80. CD•CE = AC•CB ⇒ AC:CE::CD:CB or CE:AC::CB:CD

81. Need DGFC ≡ cyclic 4-gon then 3.36.C1.

82. Need ΔAEB~ΔABD then use 6.4.

83. Proof lesson. Try ΔACF~ΔABD with 6.19 then study solution.

84. Need ∠APE = ∠FQH

85. rect∟AEDG for use as similar figure in 6.25.

86. Pure logic. How can you solve this with 6.25?

87. Solution very similar to #86.

88. seg⊙ and 6.C.

89. 6.D and Distributive Law.

90. 6.D 1.5 1.6

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Problem Solutions

The only way you will get anything out of these problems is to think about the solutions, how they are built up, and what everything means, until they become a part of your understanding. Otherwise, you are wasting your time. But you knew that.

1. $(2/3)^2 = 4/9$; $\sqrt{(100/144)} = 10/12$

2. $(3/5) \times (7/9) = 21/45 = 7/15$

3. Let 2x and 3x represent the numbers.
Then $(2x + 9)/(3x + 9) = 3/4$ and algebrate.

4. $((a+c)/(b+c))^2 = (a^2 + 2ac + c^2)/(b^2 + 2bc + c^2)$
$= (a^2 + 2ac + ab)/(b^2 + 2bc + ab)$
$= a(a + 2c + b)/b(a + 2c + b) = a/b$

5. $9 : 4 :: x+15 : x$ $\therefore 9x = 4x + 60$ and algebrate. And x is whose share?

6. $2x+4 : 3x+4 :: 5 : 7$ $\therefore 14x + 28 = 15x + 20$ and algebrate.

7. $x : x - 40 :: A : B$
$\therefore$ $2x - 40 : x - 40 :: 14 : 5$
$\therefore$ $10x - 200 = 14x - 560$ and algebrate.

8. Let's walk it through this one. The minute hand moves 12 minute marks for every 1 minute mark moved by the hour hand. So our first ratio is 12:1.
We want to know at what point after five o'clock, the hour hand moves from 5 to the same place the minute hand moves to from 12. The fractional part of this hour is equal to the minute hand's movement, numerically, though they are in the 12:1 proportion together. So $5 + x : x$ is the next ratio, improper fraction to fraction.

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We have:

$$12 : 1 :: 5 + x : x$$

$$\therefore 12x = 5 + x \text{ and } x = 5/11.$$

So the values of the second ratio are 5+(5/11) and 5/11 which is at 5:27.272727... o'clock. Counter-intuitively we have

minute rate:hour rate::hour position:minute position.

But 12 > 1 means the larger 5+x has to be over the x.

Note also that this won't work for 12 o'clock because then it would calculate from 12 to 13, the first rate would have to be $12^1/_{12}$: 1, and there would be 26 hours in a day.

It does work for 0 o'clock if you remember that 12x = 1x is not a contradiction unless you impetuously eliminate x. It works fine for x = 0 and that is exactly where the hands are: both pointing to 0 = 12.

9. A + B = 140 and A - B : A : 4 : 9 and A = x, B = 140 - x

$\therefore$ 2x - 140 : x : 4 : 9   (A-B) : A :: 4 : 9

$\therefore$ 18x - 1260 = 4x

$\therefore$ 14x = 1260 and x = 90 = A and 140 - 90 = 50 = B

10. A+B : B+C :: 5 : 4 and A = B - 60 = C + 68

$\therefore$ 2B - 60 : 2B - 128 :: 5 : 4

$\therefore$ 8B - 240 = 10B - 640 $\therefore$ 2B = 400 $\therefore$ B = 200, A = 140, C = 72

11. 9t = 4b + 90d. $\therefore$ t = (4/9)b + 10

b+3 : t :: 8 : 5 $\therefore$ b+3 : (4/9)b + 10 : 8 : 5

$\therefore$ 5b + 15 = 32/9b + 80 $\therefore$ 13b = 585 $\therefore$ b = 45d. t = 30d.

12. Let x = number of 10s. each length cost

$\therefore$ lengths are 10/x and 13/x (10s. of cloth×cost÷cost)

10/x + 10 : 13/x + 10 :: 5 : 6

$\therefore$ 10 (1/x + 1) : 3/x : 5 : 1   (a:b::c:d $\therefore$ a : b-a :: c : d-c)

$\therefore$ 2(1/x + 1) : 3/x :: 1 : 1   (a:b::c:d $\therefore$ a/5 : b :: c/5 : d)

2(x + 1)/x = 3/x $\therefore$ 2x + 2 = 3 $\therefore$ 2x = 1 $\therefore$ x = 1/2 (of 10s.)

$\therefore$ x = 5s. = cost/yard $\therefore$ lengths: 10/0.5 = 20 and 13/0.5 = 26

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Fraction Thing 1

You will be doing more algebra in life than pure geometry. So let's expand your understanding of proportion. Expressions are **homogeneous** in selected variables if these variables sum to the same powers, which is their degree of homogeneity. For example, in variables a, b, c: $3a^4$, $2a^2b^2x$, and a^2bcy are homogeneous, degree 4. We only care about the a, b, c powers. Now let a:b::c:d. If we create any two homogeneous expressions in a and b and the **same two** in c and d, proportion is maintained. So we can pull

$$2a^3 + 3a^2b \text{ and } b^3 + ab^2$$

out of our hat (all degree 3) and know that

$$2a^3 + 3a^2b : b^3 + ab^2 :: 2c^3 + 3c^2d : d^3 + cd^2$$

Since a/b = c/d = k, a = bk and c = dk. Substitute these in this last proportion and you will see how it works. In practice, you will be solving an equation's algebraic fractions. If they fit this pattern, they reduce from whatever monster they are to a simpler equation in k, which will still include the unknowns for solving the problem.

13. x:y::x+y:42 and x:y::x-y:6
∴ x+y : 42 : x-y : 6
∴ x+y : x-y : 42 : 6 (alternate)
∴ x+y : x-y : 7 : 1
∴ 2x : 2y : 8 : 6 (a+b:a-b::c+d:c-d)
∴ x : y : 4 : 3
∴ x = (4/3)y (1)
∴ 4 : 3 :: y/3 : 6 (2)
∴ y = 24 x = 32
(1) next we take x:y::4:3 and x:y::x-y:6 ∴ 4:3::x-y:6
(2) which gives us for x-y : 6 = (4/3)y - (3/3)y : 6 = y/3 : 6

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14. x-y : x+y :: 2 : 3
∴ 2x : 2y :: 5 : 1 (a+b:a-b::c+d:c-d)
∴ x : y :: 5 : 1 ∴ x = 5y
x+y : xy : 3 : 5
∴ 6y : xy : 3 : 5
∴ 2 : x :: 1 : 5 ((÷3) a/b = c/d and factor out y/y)
∴ x = 10 y = 2

15. x = gallons brandy, y = gallons water
x+6 : y+6 :: 7 : 6 ∴ x+6 : x-y : 7 : 1 (a:a-b::c:c-d)
x-6 : y-6 :: 6 : 5 ∴ x-y : x-6 :: 1 : 6 (a-b:b::c-d:d)
and x+6: x-y :: 7 : 1 (brought down for ex aeq.)
∴ x+6 : x-6 :: 7 : 6 (ex aequali from last 2 expressions)
∴ 2x : 12 :: 13 : 1 (a+b:a-b::c+d:c-d)
∴ x : 6 :: 13 : 1 ∴ x = 78 gallons brandy
∴ 84 : y+6 :: 7 : 6 ∴ 12 : y+6 :: 1 : 6 ∴ y + 6 = 72
∴ y = 66 gallons water

16. Numbers: 5x:7x and 3y:5y
5x + 3y : 7x + 5y :: 9 : 13
∴ 5x + 3y : 2x + 2y :: 9 : 4 (a:b-a::c:d-c)
∴ 5x + 3y : x + y :: 9 : 2 ∴ 10x + 6y = 9x + 9y ∴ x = 3y
2x + 2y = 16 (hyp) ∴ 6y + 2y = 8y = 16
∴ y = 2 x = 6 ∴ numbers are 30, 42, 6, 10

Fraction Thing 2

Let's do another algebraic fraction thing that could come in useful. It builds on 5.12. If a/b = c/d = e/f = k then for $\forall p,q,r \in \mathbf{R}$:

$$\left((pa^n + qc^n + re^n) / (pb^n + qd^n + rf^n) \right)^{1/n} = k \quad (\varepsilon)$$

For a = bk, c = dk, e = fk
∴ $p(kb)^n + q(kd)^n + r(kf)^n = pa^n + qc^n + re^n$
∴ (ε) is just $(k^n)^{1/n}$ = a/b = c/d = e/f

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This is true of any number of equal fractions; it just makes the (ε) fraction bigger when you have more. The p, q, r, and n can be positive or negative, integers or fractions or irrationals like e and π. You can see that Euclid's 5.12 is the case where n = 1 and p = q = r.

And I hope you have noticed how often a/b = c/d = k has come up. Simply remembering that equal things are equal to something, even if you don't know what, can be useful in algebra.

17. x = price sherry, y = price brandy both per dozen
2x + y = 3×78 = 234 (1)
7x + 2y = 9 × 79 = 711 (2)
3x = 243 ((2) - 2×(1)) ∴ x = 81s. ∴ (from (1)) y = 72s.

18. x:y is wheat to barley
10x + 4y = cost of x+y bushels
11x + 11y = selling price
∴ x + 7y = his gain (subtracting cost from sales)
∴ 10x+4y : x+7y :: 100 : 43 $^3/_4$ :: (x4) 400 : 175 :: (÷25) 16 : 7
∴ 5x + 2y : x + 7y :: 8 : 7
∴ 35x + 14y = 8x + 56y ∴ 9x = 14y ∴ x : y :: 14 : 9

19. 5x, 3x = circumferences of wheels
2x + 280 = the difference of the ropes
15x:3x (3+12:3 wraps) :: 5:1 = longer length:shorter length
So let 5y and y = lengths of longer and shorter ropes in yards
then 4y = 2x + 280 (5y - y = diff ropes)
and 5y/5x = y/3x + 12 (ropes/circumference)
∴ 2y/3x = 12 ∴ 4y = 72x
∴ 72x = 2x + 280 ∴ x = 4 ∴ y = 2
∴ circumferences are 20 and 12 yards, ropes 360 and 72 yards

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20. Line ABDC, AB = 7, AD = 20, trader's ⊥ tack CE. Join DE.
10:8::5:4 = velocity privateer: velocity trader
(from last ratios) ::AD:BC:20:BC
∴ BC = (20 × 4)/5 = 16
∴ DC = 16 - BD = 16 - 13 = 3 and DE:CE:5:4 Let CE = x
∴ $\sqrt{(9 + x^2)}$: x :: 5 : 4
∴ 9 + x^2 : x^2 :: 25 : 16
∴ 9 : x^2 :: 9 : 16 (a-b:b::c-d:d)
∴ x^2 = 16 ∴ x = ±4
∴ DE = 5 ∴ AD + DE = 25

21. Proof
E∈AD: AE = AC
∴ΔCDB = ΔCAE ∴ΔCDB : ΔACD :: ΔACE : ΔACD
∴ΔCDB : ΔACD :: AE : AD (6.1) :: AC : AB
∴ΔACD : ΔABD :: AC : AB ∴ΔCBD : ΔACD :: ΔACD : ΔABD
Note: Line 3: ::AC:AB from line 1

22. Method
AD⊥BC DK∈DA: DK = ⅓DA
BE⊥AC EL∈BE: EL = ⅓EB
KO∥BC × LO∥AC @ O O required.
Proof
ΔABC,OBC on BC, altΔOBC = ⅓altΔABC ∴ΔOBC = ⅓ΔABC
Sym. ΔOCA = ⅓ΔABC and sym. same for ΔOCB

23. Proof
CD alt∠C × AB @ D
∴ΔOBC : ΔABC :: OP : CD (6.1.C1)
ΔOCA : ΔABC :: OQ : CD
ΔOAB : ΔABC :: OR : CD
∴ ∑ΔO[BC,CA,AB] : ΔABC :: ∑O[P,Q,R] : CD
∑ΔO[BC,CA,AB] = ΔABC ∴ ∑O[P,Q,R] = CD
Note: It's important to recognize equalities in proportions.

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24. Proof

ΔAFE = ΔFDE (1.34) ΔFDE = ΔFDC (1.37) ∴ΔAFE = ΔFDC

∴ΔBFD : ΔAFE :: BD : DC (6.1)

ΔAFE : ΔEDC :: AE : EC (6.1) :: BD : DC (6.2)

∴ΔBFD : ΔAFE :: ΔAFE : ΔEDC

Note: In my diagram, this amounts to 1 : 1 :: 1 : 1, which is true...

25. Proof

CF : FB :: AE : EB (6.2) DG : GB :: AE : EB (6.2)

∴CF : FB :: DG : GB ∴FG∥CD (6.2)

Note: 6.2's proportion is how we say "parallel" using proportions.

26. Proof

O mdpt CK (1.34) POQ∥AB × AC,BC @ P,Q ∴CP:PA::CO:OC (6.2)

CO=OK ∴CP=PA ∴P mdpt AC Sym. Q mdpt BC

∴O∈ line of mdpts of AC,BC

Nore: The game has really changed. Learn to think in proportions. Here they are used to prove equality. What were they used for in #21 through #25?

27. Proof

AD × BC @ F ∴AD=DF (1.26)

AD:DF::AE:EC (6.2) ∴AE=EC

28. Proof

ΔBED = ΔCED (1.37) ∴ΔDFB = ΔEFC

ΔADF : ΔBDF :: AF : BF (6.1) ::AE:EC (6.2)

:: ΔAEF : ΔEFC (6.1) :: ΔAEF : ΔBDF ∴ΔADF = ΔAEF

Note: Don't let your initial ignorance of how one uses Book V get to you. Just start figuring it out by studying its usage in these solutions. Get some light going in the darkness.

29. Proof

AF(pr) × BC @ H ∴ΔBFH : ΔBFA :: FH : FA (6.1) :: ΔCFH : ΔCFA

ΔAFB = ΔAFC (#28) ∴ΔBFH : ΔBFA :: ΔCFH : ΔBFA

∴ΔBFH = ΔCFH ∴BH=CH

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30. Proof

CGH∥DA × EF,AB @ G,H ∴ CE,GA ≡ ∥gm ∴ CG=DE GH=EA
CG:GH::CF:FB (6.2) ∴ DE:EA::CF:FB

31. Method

Q mdpt PA QR∥AC × BC @ R PR × AC(pr) @ S PS required

Proof

PR:RS::PQ:QA (6.2) PQ=QA ∴ PR=RS ∴ BC ×/2 PS @ R

32. Proof

AF:FC::AD:DC (6.3) AE:EB::AD:DC (6.3)
∴ AF:FC::AE:EB ∴ EF∥BC (6.2)

33. Proof

B mdpt arcDBC (3.30) ∴ GE ×/2∠DGC (3.27) ∴ DG:GC::DE:EC (6.3)
Sym. DF:FC::DE:EC ∴ DG:GC::DF:FC ∴ DG:DF::GC:FC (5.16)

34. Proof

PA ×/2∠CPD ∴ CA:AD::CP:DP (6.3)
∠APB = ∟ (3.31) ∴ PB ×/2ext∠P ∴ CB:DB::CP:PD (6.A)
∴ CA:AD::CB:BD ∴ AC:BC::AD:BD

35. Method

∀magnitudeM: ⊙A,M × ⊙B,2M @ C.C' Join C[A,B]
CD ×/2∠ACB × AB @ D E mdpt DB D,E required

Proof

AC:CB::AD:DB BC=2AC ∴ DB=2AD ∴ AD=DE=EB

36. Method

⊙diamAB C mdpt arcAB CD(pr) × ⊙ @ E
PE⊥CE × BA(pr) @ P P required

Proof

arcAC = arcBC ∴ ED ×/2∠AEB (3.27) ∴ AE:EB::AD:DB (6.3)
∠DEP = ∟ ∴ EP ×/2ext∠E ∴ AE:EB::AP:BP (6.A) ∴ AD:DB::AP:BP

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37. Proof

AB=AE (hyp) ∴ ∠AEB = ∠ABC

∠EAD = ∠BAC ∴ ED=BC (1.26)

AC ×/2∠BAD AE⊥AC ∴ BC:CD::BA:AD (6.3) BE:ED::BC:CD (6.A)

ED=BC ∴ BE:BC::BC:CD

Note: The past few problems have been Todhunter's way of introducing the diagrams of 6.3 and 6.A into various settings. This is how **most** problems are created.

38. Proof

BD:DC::BA:AC (6.3) ∴ BD-DC:BD+DC::AB-AC:AB+AC

BD-DC = 2DO BD+DC = 2BO

∴ 2DO:2BO::DO:BO (5.15) ∴ DO:BO::AB-AC:AB+AC

Note: Line 2 is an example of **using** the diagram.

39. Proof

BD:DC::BE:EC (6.3,A) ∴ BD-DC:DC::BE-EC:EC

∴ 2OD:DC::2OC:EC ∴ OD:DC::OC:EC

∴ OD:OD+DC::OC:OC+EC ∴ OD:OC::OC:OE

OC=OB ∴ OD:OB::OB:OE

40. Proof

AD × FE @ G DE(pr) to ∀H ∠GEA = ∠DEC ∴ AE ×/2∠GEH

∴ DE:EG::DA:GA Sym. DF:FG::DA:GA ∴ DE:EG::DF:FG

∴ DE:DF::EG:FG ∴ DG ×/2∠FDE (6.3) ∴ ∠GDF,FDB = ∠GDE,ECD

∴ ∠GDB = ∠GDC = ∟ ∴ AD⊥BC Sym. other 2 ∠

41. Proof

KL:AB::CL:CB MN:DE::FM:FD (6.4)

∴ CL:CB::FM:FD (#30)

∴ KL:AB::MN:DE ∴ KL:MN::AB:DE

AB=DE (hyp) ∴ KL=MN ∴ ΔCKL = ΔFMN (1.38)

42. Proof

CE:AB::FE:FB ED:AB::GD:GB (6.4)

CE=ED ∴ FE:FB::GD:GB ∴ FG∥DE (6.2)∥AB

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43. Proof

BN:AM::CB:CA ∴ BN:AM constant

Note: Yes, it's that easy. BC and AC are fixed.

44. Proof

AM:AC::BN:BC (6.4) ∴ AM:BN::AC:BC

AM:BN fixed ∴ AC:BC fixed ∴ C fixed

45. Method/Proof

∀D,E∈AC,BC AF,CG,BH⊥DE ∴ AF:CG::AD:DC (6.4)

AF:CF given ∴ D fixed Sym. E fixed ∴ correct DE given

Note: You couldn't easily construct DE. But the proof shows that from any D,E you can find the required D,E.

46. Method/Proof

AC(pr) to D: AC:CD = given ratio Join BD ECF⊥BD ECF required

AM⊥EF EF × BD @ N ∴ CM:CN:AC:DC (6.4)

47. Proof

∠BFD = ∠CFE (1.15) ∠DBF = ∠FEC (1.29)

∴ΔBFD eq∠ΔEFC (1.32) ∴ BD:CE::BF:FE

BD=BA CA=CE ∴ BA:AC::BF:FE ∴ AF∥CE(6.2)∥BD

48. Proof

AC × BD @ O ∠DOC = ∠BOA (1.15) ∴ΔDOC~ΔBOA

∴ AO:CO::AB:CD (6.4) :: 2:1 ∴ OC = ⅓AC OD = ⅓BD

49. Proof

OE∥AB × BC @ O ∴ AB:AC::OE:EC (6.4) ::OE:BD (hyp) ::EF:DF (6.4)

50. Proof

1) PY,QZ⊥AB,CD ∴ΔPMY~ΔQNZ

∴ PM:QN::PY:QZ ∴ PM:QN constant

2) NM × PQ @ R ∴ΔPMR~ΔQNR ∴ RP:RQ::PM:QN

∴ RP:RQ constant ∴ R∈∀MN

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51. Proof

∠CAT = ∠CBT = ∟ ∴ CBAT cyclic 4-gon
∴ ∠CAB = ∠CTB (3.21)
∴ ∠ABN = ∠CTB ∟ANB = ∟CBT ∴ ΔBAN~ΔBCT (1.32)
∴ BT:BC::BN:NA (6.4)

52. Proof

∠APC = 2(2∟-∠ADC) (3.20, 3.22) ∴ ∠APC = 2∠ADB
∠AQB = 2∠ADB (3.20) ∴ ∠AQB = ∠APC
∴ isosΔAQB~isosΔAPC ∴ PA:QA::AC:AB
Note: Or diameters here are proportional to sides of ΔABC.

53. Proof

PQ × CA @ L ∴ LP:LQ::AP:CQ (6.4)
PQ × BD @ M ∴ MP:MQ::PB:QD
AP:PB::DQ:QC (hyp) ∴ LP:LQ::MP:MQ ∴ L ≡ M
Note: Sym. PQ,AD,BC concur. You will need this method again.

54. Proof

CE∥AB × BD @ E C mdpt AD ∴ E mdpt BD (6.2)
ΔMAP~ΔDCE ∴ MP:PA::DE:EC (6.4)
ΔNBQ~ΔBCE ∴ NQ:QB::BE:EC (6.4) :: DE:EC
∴ MP:PA::NQ:QB ∴ MP:NQ::PA:QB
MP=NQ (1.34) ∴ PA=QB

55. Proof

ΔAEM~ΔAFM ∴ AM:AN::AE:AF::AB:AC ∴ AM:MP::AB:BD
∴ ΔAMP~ΔABD (6.6) ∴ P∈AD
Note: If EB,AF homologous, P∈BC.

56. Proof

GE(pr) × CA(pr) @ P ∴ CP:KP::CG:KE (6.4)
FH(pr) × CA(pr) @ Q ∴ CQ:KQ::CF:KH (6.4)
ΔCGK~ΔKEA ∴ CG:GK::KE:EA ∴ CG:CF::KE:KH
∴ CG:KE::CF:KH ∴ CP:KP::CQ:KQ ∴ P ≡ Q

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57. Method/Proof

line∥BC @ distance X × line∥CA @ distance Y @ D
∴ ⊥from D on BC : ⊥from D on CA = X:Y Join CD
∀P∈CD: PM,PN⊥BC,CA ∴ PM:X::CP:CD::PN:Y (6.4)
∴ PM:PN::X:Y **(for ∀P∈CD)**
Sym. w/lines∥CA,AB, ∃!E: ⊥from E on CA:⊥ from E on AB = Y:Z
∴ ∀P∈AE: PM:PN = Y:Z ∴ CD × AE @ O
∴ ⊥ on O to BC,CA,AB in ratio X:Y:Z
Note: Make sure you understand this one.

58. Proof

RS × ⊙C @ S,P ×BC(pr) @ T
∠CPS = ∠CSP (1.5) = ∠ASR (1.15) = ∠ARS (1.5)
∴ AB∥CP ∴ TC:TB::CP:BR (6.4) ∴ TC:TB fixed ∴ T fixed
Note: Ratios show fixed point by have a known fixed ratio (CP:BR) proportional to a ratio with the point (TC:TB).

59. Proof

ΔAOE,AED: ∠EAO = ∠EAO ∠AEO = ∠ADE (4.14, 3.27)
∴ ΔAOE~ΔAED (1.32) ∴ AO:AE::AE:AD

60. Proof

QR:BR::PQ:AB (6.4) ::PQ:DC (1.34) ::SQ:CS (6.4)
∴ QR:BR::RS:BC ∴ RS∥BC(6.2)∥AD
Note: ∥ in Δ cuts sides proportionally in 6.2. So, conversely, we use the proportions from 6.2 to show ∥. But you knew that.

61. Proof

EM,FN∥AD × BC @ M,N
EM:MC::AD:DC (6.4) ::AD:DB (hyp) ::FN:NB (6.4)
∴ EM:FN::MC:NB
EM:MB::PD:DB (6.4) ::PD:DC (hyp) ::FN:NC (6.4)
∴ EM:FN::MB:NC
∴ MC:NB::MB:NC ∴ MC:MB::NB:NC ∴ MC:CB:NB:CB ∴ MC=NB
EM:FN::MC:NB ∴ EM=FN ∴ FE∥NM (1.33) ∴ EF∥BC

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62. Proof

SL:SP::TL:TQ (6.4) ∴ SL-SP:SP::TL-TQ:TQ (5.17) ∴ EL:SP::BL:SQ
SP = SE = 3ET = 3TQ ∴ EB = 2BL ∴ BL=BT

63. Proof

BD:EF::CD:CF EF:AC::FD:CD (6.4) ∴ BD:AC:FD:CF (5.23)
∴ BD:FD::AC:CF ∴ ΔBDF~ΔACF ∴ ∠BFD = ∠AFC

64. Proof

ΔOAE~ΔOBF ∴ OA:OE::OB:OF ∴ ΔOEF = ΔOAB ∴ ∠OFE = ∠OBA
Sym. ∠OEH = ∠OCB = ∠BDA ∴ ∠FEH = ∠BAD
∠EFH = ∠ABD (proven) ∴ ∠FHE = ∠BDA ∴ ΔFEH~ΔBAD
Sym. ΔFGH~ΔBCD ∴ ‖EFGH~‖gmABCD

65. Proof

BF × CD @ F ∴ FB=FC=FD (v2#18) ∴ B,C,D∈⊙F,FB ∴ ∠CBD = ∟
∴ ∠EBD = ∟ (3.31) ∴ CBE colinear (1.14) Sym. DBA colinear
∠BCD = ∠CAB ∠BDC = ∠DEB (3.32) ∴ ΔACD~ΔCDE
∴ ED:DC::DC:CA

66. Method

F mdpt arcEGK H∈chordEK: EH:HK = given ratio
FH(pr) × arcEDK @ L required

Proof

arcEF = arcFK ∴ ∠ELF = ∠FLK (3.27) ∴ EL:LK::EH:HK (6.3)

67. Proof

CA:CP:CP:CB ∴ ΔACP~ΔPCB (6.6) ∴ ∠CPA = ∠CBP
Note: Given the proportion of 6.6, you have two similar Δ from the points in the two ratios. But make sure you get the orientation right.

68. Proof

BD:BA:BA:BC ∴ ΔBDA~ΔBAC (6.6) ∴ ∠BAC = ∟BDA

Have you noticed how often you use 6.4 then hyp or con and then 6.4 again? Doing this is one of your **tools**.

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69. Proof

∠BDA = ∠CDA = ∟ BD:AD:AD:CD ∴ΔBDA~ΔADC (6.6)
∴ ∠BAD = ∠ACD ∠ABD = ∠CAD
∴ ∠BAC = ∠BAD + ∠DAC = ∠B + ∠C ∴ ∠A = ∟

70. Proof

∠SCT = ∟ (v2#30) ∴SP:CP::CP:PT (6.8.C1)
∴SP•PT = CP^2 (6.17) = $radius^2$ ∴SP•PT constant

71. Method

CE∥AB: CE:CB::CB:BA BE × AC @ D DF∥AB × BC @ F required
Proof
DF:FB::CE:CB (6.4) ::CB:BA (con) ::CF:FD (6.4)
∴DF:FB::CF:FD ∴FB:DF::DF:CF (5.B)
Note: The required line is the mean proportional between its segments of the base.

72. Proof

ΔECA eq∠ ΔBCD ∴EC:CA::BC:CD (6.4) ∴ΔECD = ΔACB (6.15)
Note: From 6.15 proportion, triangle from points of extremes equals triangle from points of means.

73. Proof

ΔABE eq∠ ΔCBF ∴AB:BE::CB:BF (6.4)
∴AB:CB::BE:BF ∴ΔABF = ΔCBE (6.15)

74. Proof

∠AOD = ∠BOC (1.15) ∠DAO,ADO = ∠CBO,BCO (3.21)
∴ΔAOD~ΔBOC Sym. ΔCOD~ΔBOA
Note: From ΔAOD~ΔBOC ∴DO:AO:CO:BO (6.4)
∴DO•BO = AO•CO (6.16) which proves 3.35 if you think about it.

75. Proof

EF × CD @ M ∴GO:EM::CO:CM LO:FM::CO:CM (6.4)
∴GO:EM::LO:FM Sym. KO:HO::EM:EM
∴GO:LO::KO:HO ∴GO•HO = LO•KO (6.16)

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76. Method/Proof

Assume D: ∆ADE = ∆DBC

∴ AD:DB::BC:DE (6.15)

BC:DE::AB:AD (6.4) ∴ AD:DB::AB:AD ∴ DB:AB = AD^2 (6.16)

∴ take D∈AB by method of 2.11

Note: Proof is by analysis using pure reason. If you know 2.11, you don't even need a diagram.

77. Proof

∆ADF~∆GEB ∴ EB:EG::DF:DA ∴ AD•EB = EG•DF (6.17) = DE^2

78. Proof

∠ABD = ∠AEC (3.21) ∠ADB = ∠ACE (con) ∴ ∆BAD~∆EAC (1.32)

∴ AB:AD::AE:AC ∴ AC•AB = AD•AE (6.17)

79. Proof

∆AFE~∆CFB ∴ EF:FB::FA:FC (6.4)

∆GFC~∆BFA ∴ BF:FG::FA:FC (6.4)

∴ EF:FB::BF:FG ∴ EF•FG = BF^2 (6.17)

80. Proof

AC:CE::CD:CB (hyp) ∠ACE = ∠DCB ∴ ∆ACE~∆DCB (6.6)

∴ ∠CEA = ∠CBD ∴ CBEA ≡ cyclic 4-gon

∴ E mdpt arcAEB ∴ ∠C,AB fixed ⇒ ⊙CBEA fixed ∴ E fixed

Note: Learn how to unwind 6.17. See hint for this problem. Also, figure out why those two things in line 1 make the ∆s similar by 6.6 and why ∠CEA=∠CDB makes the cyclic 4-gon.

81. Proof

∠DFC + ∠ACD + ∠BDC + ½∠ACB = 2∟ (1.32)

∠DGC + ∠ACD + ∠BDC + ½∠ADB = 2∟

∴ ∠DFC + ½∠ACB = ∠DGC + ½∠ADB

∠DFC = ∠DGC (3.21) ∴ DGFC ≡ cyclic 4-gon

∴ EG•ED = EF•EC (3.36.C1) ∴ EF:EG::ED:EC

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82. Proof

$\angle AEB = \angle ACB$ (3.21) $= \angle ABC$ (1.5) $\therefore \Delta AEB \sim \Delta ABD$ (1.32)
$\therefore DA:AB::AB:AE$ (6.4) $\therefore DA \bullet AE = AB^2$ (6.17)

83. Proof

$\Delta ACF \sim \Delta ABD \therefore \Delta ACF : \Delta ABD : AC^2 : AB^2$ (6.19)
$:: AB \bullet BC : AB^2$ (4.10) $:: BC : AB \therefore \Delta ACF : BCFD :: BC : AB$ (5.C)
$AC = BD \therefore \Delta ACF : BCFD :: BC : BD$
$\Delta BCD \sim \Delta BAD \therefore \Delta ACF : BCFD :: BD : BA$

84. Proof

$\Delta AEF \sim \Delta FHC \therefore AE:AF::FH:FC$
$AP,FQ = \frac{1}{2}AF, \frac{1}{2}AC \therefore AE:AP::FH:FQ$
$\therefore \Delta AEP \sim \Delta FHQ$ (6.6) $\therefore \angle APE = \angle FQH \therefore EP \parallel HQ$ (1.28)

85. Method/Proof

$CH \perp AB$: rect$\llcorner AFCH$
rect$\llcorner AEDG$: $AEDG \sim AFCH$ $AEDG = \Delta ABC$ (6.25) $E,G \in AH,AF$
$\therefore D \in AC$ (6.26) $\therefore AED = \frac{1}{2}AEDG = \frac{1}{2}\Delta ABC \therefore ED$ required.

86. Method/Proof

$\forall$chordCBD: $CB \bullet BD$ known $\therefore CB:BD$ known $\therefore CB \bullet BD$ by 6.25
$\therefore$ chord CBD from last step required
Note: Eventually, you will realize that when a problem is sufficiently vague, its solution is a more general one like this.

87. Method/Proof

$\forall BE$ tan to $\odot A \Rightarrow \forall$secantBCD: $BC \bullet BD = BE^2$
$BD = 2BC \therefore BC \bullet 2BC$ by 6.25 $\Rightarrow$ secantBCD required.
Note: $BC \bullet BD = 2BC^2 = BE^2$ So to **construct** an answer would be quite a task for Euclid. Numerically, $BC = BE/\sqrt{2}$

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88. Method/Proof

seg⊙ on BC = ∠A ∴ diam⊙BC known

∴ magnitudeAD alt∠A is known (6.C)

EF‖BC @ distance AD from BC × seg⊙ @ A,A'

ΔABC or ΔA'BC required

89. Proof

APCB ≡ cyclic 4-gon ∴ PB•AC = PA•BC + PC•AB (6.D)

AC=AB=BC ∴ AC•PB = AC(PA + PC) ∴ PB = PA + PC

90. Proof

∠ABD = ∠ACD = ∟ ∴ ABCD ≡ cyclic 4-gon

∴ AD•BC = AB•CD + AC•BD (6.D)

AB=AC ∴ ∠ABC = ∠ACB (1.5) ∴ ∠DBC = ∠DCB ∴ DB=DC (1.6)

∴ AD•BC = AB•DB + AB•DB = 2(AB•DB)

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Notation

Labelling is done top to bottom, left to right; or clockwise from top-left apex of non-triangular figure. Labelling in propositions follows that of the original 1867 diagrams.

Operators

intersect, cut	×
bisect, bisector	×/2
trisect	×/3
at	@
parallel	‖
between	·|·
A between B and C	A·|·(B,C)
perpendicular	⊥
AB perpendicular to CD	AB⊥CD
equivalent, equal in every way	≡
equal in magnitude	=
on	∈
not on	∉
equilateral (equal sides)	eqS
equiangular	eq∠
equidistant	eqD
distance from A to B	D(A,B)
absolute difference	~
|a-b|	~(a,b) or a~b
summation	∑
A+D+C+D	∑[A-D]

Points

on or endpoints of lines	A, B, C, ...
considered in themselves	P, R, S, ..
as center of a figure	O

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Lines

by endpoints	AB
creation from points	Join AB
Join AB, AC, AD	Join A[B-D]
mid-point	mdpt
P mdpt AB, Q mdpt CD	P,Q mdpt AB,CD

Angles

angle	∠
interior angle	int∠
exterior angle	ext∠
alternate angle	alt∠
opposite angle	opp∠
right angle	∟

Triangles

triangle	∆
right triangle	⊿
∀triangle	∆ABC
equilateral triangle	eqS∆
equiangular triangle	eq∠∆
isosceles triangle	isos∆
CF bisector of angle C	CF ×/2 ∠C
AD median on angle A	AD med∠A
BE altitude on angle B	BE alt∠B

Circles

circle	⊙
create by center and radius	⊙A,AB
as existing circle	⊙A
as defined by three points	⊙ABC
touching center	on center
on circumference	∈⊙
in circle's whitespace	in ⊙
line PQ tangent @ point A	tangent PAQ

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circumcircle	en⨀
inscribed circle	in⨀
escribed circle	ex⨀

Polygons

polygon	n-gon
by number of sides (4+)	4-gon
parallelogram	‖gm
rectangle	rect∟
rectangle, sides AB,CD	AB•CD
square on line AB	AB^2

Logic

therefore	∴
symmetrically	Sym.
by hypothesis	(hyp)
by construction	(con)
contradiction	⮯
any, every, each, all	∀
exists, exists only one	∃ , ∃ !
not, not equivalent	! !≡
if and only if	iff

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Euclid's Axioms, Postulates, and Definitions

All of the following are from Loney's last edition of Todhunter's Euclid. Their numbering differs slightly from another version of Todhunter's. And looking around, there is no conclusive numbering. All are close. Beyond that, you will find that there is a bit of back and forth between axioms and postulates from text to text as well. Corollaries date from the 17thC and can differ from text to text. **The numbering of the propositions is Euclid's and is the same in all Euclid texts.**

Euclid's Axioms
Book I
a.1 Things equal to the same thing are also equal to one another.
a.2 Things added to equals make equals.
a.3 Things taken from equals leave equals.
a.6 Things twice the same thing are equal to each other.
a.7 Things half of the same thing are equal to each other.
a.8 The whole is greater than its part.
a.9 Magnitudes that can be made to coincide are equal.
a.10 Two lines cannot enclose a space. They must have 0, 1, or all points in common.
a.11 All right angles are equal.
a.12 If a line cut two other lines such that, on one side of the first, the other two make angles summing to less than two right angles, the lines, extended on that side, must intersect.

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Book V

a.5.1 Equimultiples of the same or equal magnitudes are equal to each other.

a.5.2 The magnitudes of which the same or equal magnitudes are equimultiples are equal to one another.

a.5.3 A multiple of a greater magnitude is greater than the same multiple of a lesser magnitude.

a.5.4 The magnitude of which a multiple is greater than the same multiple of another is greater than that other magnitude.

Euclid's Postulates

p.1. A line may be drawn between any two points.

p.2. A line may be indefinitely extended.

p.3. Any point and any line from it may be used to construct a circle.

Euclid's Definitions

Book I

d.1.1 A **point** is position without magnitude.

d.1.2 A **line** is length without breadth.

d.1.3 The **extremities** and **intersections** of lines are points.

d.1.5 A **surface** is length and breadth.

d.1.6 The **boundaries** of surfaces are lines.

d.1.7 A **plane** is a surface such that, for any two points, their line lies entirely on the surface.

d.1.8 A **plane angle** is the inclination of two lines to one another which meet on the plane.

d.1.9 A **plane rectilinear angle** is the plane angle of two straight lines which meet at their **vertex**.

d.1.10 When a line meets another so that the two angles created by the former on one side of the latter are equal, these are **right angles** and the lines are **perpendicular**.

d.1.11 An **obtuse angle** is greater than a right angle.

d.1.12 An **acute angle** is less than a right angle.

d.1.13 A **plane figure** is any shape enclosed by lines, which are its

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perimeter.
d.1.15 A **circle** is a plane figure bounded by its **circumference,** which is equidistant from its **center**.
d.1.20 A **triangle** is bounded by three straight lines. Any of its angular points can be its **apex** which is opposite its **base**.
d.1.22 A **polygon** or **n-gon** is a plane figure with n lines for sides. A figure with 4 sides is a 4-gon or "quadrilateral."
d.1.23 An **equilateral triangle** has three equal sides.
d.1.24 An **isosceles triangle** has two equal sides.
d.1.29 **Parallel lines** are coplanar lines which cannot be produced to intersect.
d.1.30 A **parallelogram** is a 4-gon of opposing parallel sides.
d.1.31 A **square** is an eqS 4-gon with one right angle.
d.1.33 A **rhombus** is an eqS 4-gon with no right angles.

Book II
d.2.1 ∀rectangle ABCD is **contained** by any two adjacent sides. In our notation, this is "rectangle ABCD ≡ AB•AD"
d.2.2 In a ‖gm, there are two internal ‖gms on a diagonal and two complements. The complements combined with either internal ‖gm is a **gnomon**.
d.2.3 ∀AB produced in both directions: if we choose a point (cut) between A and B, we divide AB **internally**. If we choose a point to either side, outside of AB, we divide AB **externally**.

Book III
d.3.1 **Equal circles** (⊙) have equal radii, therefore equal diameters.
d.3.2 A line **touches** a ⊙ if it meets the ⊙ and, produced, does not cut it. This is a **tangent (tan)** with its **point of contact**.
d.3.3 ⊙s **touch** when they meet but do not cut each other. If ⊙A is in ⊙B they touch **internally**, else **externally**.
d.3.4 A line cutting a ⊙ at two points is a **secant**.
d.3.5 A **chord** is a line joining two points ∈⊙. A secant produces a chord.
d.3.6 Chords are **equidistant** (eqD) from ⊙ center when their

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perpendiculars (⊥) from their midpoints to ⊙ center are equal. Of two chords, the one with the greatest ⊥ is **farther** from center.
d.3.7 A **segment** of a ⊙ is a chord and what it cuts off, away from ⊙ center. Segments of circles are **similar** if their angles are equal.
d.3.8 A **segment's angle** is contained by any point ∈⊙ joined to the endpoints of its chord. This gives a segment two angles.
d.3.9 Any part of a ⊙'s circumference is an **arc**.
d.3.10 A **sector** of a ⊙ is bounded by two radii and the arc between them.
d.3.11 ⊙s with same center are **concentric**.

Book IV
1. An n-gon is **inscribed** in another n-gon when every vertex of the first n-gon is on the side of the second.
2. An n-gon is **described** on another n-gon when every vertex of the second n-gon is on a side of the first.
3. An n-gon is **inscribed in a ⊙** when all of its vertices lie on the ⊙.
4. An n-gon is **described on a ⊙** when all of its sides are tangent to the ⊙.
5. A ⊙ is **inscribed** in an n-gon when all sides of the n-gon are tangent to the ⊙.
6. A ⊙ is **described** on an n-gon when all vertices of the n-gon lie on the ⊙.
7. A line is **placed in a ⊙** when it is made a chord of the ⊙.

Book V
d.5.1 A lesser magnitude is a **part** or **submultiple** of a greater when the lesser measures the greater exactly.
d.5.2 A greater magnitude is a **multiple** of a lesser when the lesser measures the greater exactly.
d.5.3 **Ratio** is the comparison of two magnitudes of the same kind: length, area, arc, etc.
d.5.5 When two ratios, a:b and c:d have the **same ratio** they are expressed as a:b::c:d

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d.5.6 Magnitudes which have the same ratio are called **proportionals** and expressions like a:b::c:d are **proportions**.
d.5.9 Proportions consist of at least three members.
d.5.10 If a:b::b:c, then a:c is the **duplicate ratio** of a:b.
d.5.11 If a:b::b:c::c:d, then a:d is the **triplicate ratio** of a:b and so on. This kind of proportion is called **continued proportion**.
d.5.12 Of any number of continued proportionals, the first has to the last the **compound ratio** of the series.

Book VI

d.6.1. **Similar** rectilineal figures are equiangular and have proportional sides about those angles.
d.6.2. Two **reciprocal** triangles or parallelograms (A,B) have sides about their angles such that

side 1 of A : side 1 of B :: side 2 of B : side 2 of A

d.6.3. A line is cut into **extreme and mean ratio** when

whole : greater segment :: greater : lesser

d.6.4. The **altitude** of a figure is the line from its vertex, perpendicular to its base.

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