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DIVIDING A SEGMENT IN A GIVEN RATIO

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ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

Н.М. Бескин

ДЕЛЕНИЕ ОТРЕЗКА В ДАННОМ ОТНОШЕНИИ

Издательство «Наука»

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DIVIDING A SEGMENT IN A GIVEN RATIO

Translated from the Russian
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MIR PUBLISHERS
MOSCOW

First published 1975
Revised from the 1973 Russian edition

На английском языке

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PREFACE

Some pupils have fine appetite for mathematics. They are not satisfied with the portions of mathematics metered by the school curriculum. Where should they look for some more?

Their mathematical knowledge can be extended in breadth or in depth. In breadth—by studying new branches of mathematics. In depth—by a more thorough analysis of the problems, comprised by the school curriculum. There is no branch of mathematics such that one could have the right to say: "I have complete knowledge of this". The most elementary problem has hidden in it unexpected connections with other problems, and this process of going deeper and deeper into a problem has no end. We can return to a familiar branch time after time and (if we think thoroughly) each time we learn something new.

This booklet will lead the reader into depth. By analyzing a very elementary problem of how to divide a line segment in a given ratio we shall learn many new things.

The problem as such will be taken up in Chapter I. The introduction contains some technical data which are necessary to develop the principal subject.

Introduction

1. Orientation of a straight line and a line segment. There are two different directions on a straight line. *To ascribe an orientation* to a straight line is to choose one of the two. A straight line for which one of the two orientations has been chosen is called *an orientated line* or *an axis.*

In the following we shall always use the phrase "straight line" to mean a non-orientated straight line. On this line the two directions are equivalent.

In a figure the direction chosen is usually marked by an arrow (Fig. 1). One can say that an axis is a pair, formed by the following two elements: (1) a straight line, (2) one of the two possible directions on it.

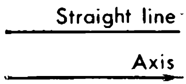


FIG. 1

A line segment is a part of a straight line bounded by two points. These points are *the ends of the segment* (and belong to it too). The ends of the segment can be *ordered*, i.e. one can take one of them to be the first and the other the second. Usually the former is called *the initial point* of the line segment and the latter *the end point*, or simply *the end.* A segment with the ordered ends is called an *orientated* segment. In order to show in a figure that the segment is orientated its ends should be marked differently, say, by different letters, or by an arrow at one of the ends, etc. One can say that an orientated line segment is a pair, formed by the following two elements: (1) a segment, (2) one of its two ends (that which is taken to be the first end or the initial point).

If a segment is marked by two letters and is not orientated the sequence of the letters is arbitrary: AB or BA is the same line segment. If the segment is orientated the letter at the initial point is placed first and that at the end—the second. In this case AB and BA are different line segments (they differ in their orientation).

2. Directed line segments. An orientated line segment on an axis is called *a directed line segment.*

In Fig. 2*a* the line segment AB (A is the initial point, B the end point) is not a directed segment: it is orientated but the straight line on which it lies is not an orientated one.

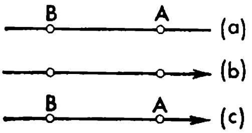


FIG. 2

The segment in Fig. 2*b* is not directed either (it is not orientated itself). The line segment AB in Fig. 2*c* is a directed segment.

It follows that to specify a directed line segment we have to give two orientations: (1) of the segment itself, (2) of the straight line on which it lies. The two orientations are given independently, i.e. each of them may be given in any of the two possible ways.

Every line segment has a length. The length is a non-negative number. It can be zero only in case the ends of the segment coincide, i.e. if the segment degenerates into a point; for any non-degenerated segment its length is strictly positive. The length of the line segment AB will be designated by the symbol $\overline{AB}$. When determining the length of a segment its orientation is of no consequence.

A sign can be ascribed to a directed segment, besides its length, according to the following rule: *a directed segment is considered to be positive (negative) if its direction*[1] *coincides (does not coincide) with the direction of the axis.*

If a segment, though orientated, lies on a non-orientated straight line no sign can be ascribed to the segment.

Thus, directed segments are expressed by real numbers, positive and negative. For instance, the notation

$$AB = -3$$

means that: (1) the length of the line segment AB is 3, (2) the direction of the segment AB is opposite to the direction of the axis on which it lies (does not coincide with the latter direction).

---

[1] The orientated segment is taken to be directed from the initial point towards the end.

Here the symbol AB designates simultaneously a geometric figure (a directed line segment) and the number which corresponds to it.

Practice shows that this use of the symbol does not involve misunderstandings. It is even permissible to use the following expression: "the directed line segment is equal to —3".

If A, B, C are any three points lying on an axis, then

$$AB + BC = AC. \tag{2.1}$$

This equality is called *the chain rule* or *Chasles' formula*. It has a deep meaning worth being thought about. If AB, BC and AC denoted the lengths of line segments, formula (2.1) would hold true if and only if point B is situated *between* A and C. However, if we are dealing with directed segments, formula (2.1) holds with any relative positions of points A, B, C. Accordingly it is applicable without specifying any conditions or looking at the drawing. We have only to remember the order of the letters in the symbols of this formula.

The proof of formula (2.1) is obtained easily by considering all the possible arrangements of point B with respect to segment AC.

According to (2.1) a segment PQ on the axis can be divided by any point X on the same axis so that

$$PQ = PX + XQ.$$

Formula (2.1) can be generalized to read

$$AB + BC + CD + \ldots + KL + LM = AM. \tag{2.2}$$

Formula (2.2) is called *the general chain rule*. Its proof is easily obtained by a sequence of contractions: AC is substituted for $AB + BC$, then AD for $AC + CD$, etc.

It is clear that the orientation of a directed line segment is changed, when the letters in its symbol are interchanged and therefore its sign is changed (the absolute length remains unchanged):

$$BA = -AB. \tag{2.3}$$

Formula (2.3) can be obtained in a purely formal way by substituting in (2.1) letter A for C.

Using directed line segments one can introduce coordinates on the axis. In order to do it we must select on the axis *the origin of the coordinates* O and *a scale unit*.

If A is a point on the axis, the ratio of the directed line segment OA to the scale unit e is a coordinate (or abscissa) of point A

$$x = \frac{OA}{e}. \tag{2.4}$$

Two important points should be stressed. First, no sign is ascribed to the scale unit e (i.e. we always take it to be positive). It follows that the sign of the abscissa x coincides with that of the directed segment OA. Secondly, the coordinate is dimensionless, i.e. it is an abstract number. In Fig. 3 the coordinate of point A is 3 (its sign is +).

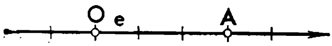


FIG. 3

Let two points on the axis be given by their coordinates: A (x_1) and B (x_2). What is then the expression for the directed segment AB?

It is undesirable to use a drawing in finding the answer to this question, because we shall then have to consider many different cases (which of the coordinates is the greater one, what are their signs, what is the position of the origin O with respect to the segments AB?). The following simple calculation solves the above problem:

$$AB = AO + OB = -OA + OB = x_2 - x_1$$ [1].

Thus, always

$$AB = x_2 - x_1. \tag{2.5}$$

Note: *a directed line segment is equal to the coordinate of its end point less the coordinate of the initial point.*

Last, let us note one more property of directed segments. If $AB = AC$, then point C coincides with point B. Expressed in symbols this statement is written as follows

$$(AB = AC) \Rightarrow (C \equiv B) \tag{2.6}$$

($\Rightarrow$ means "it follows", and $\equiv$ is the symbol for identity).

---

[1] Here and in the following we take e to be a unit segment, i.e. we take it to have a length equal to unity.

CHAPTER I

Simple Ratio

3. Statement of the problem. In order to solve any problem successfully it is necessary first of all to formulate it precisely.

The formulation "dividing a segment in a given ratio" is vague in many respects. What segment—orientated or not? Does it lie on an axis or on a straight line? What is meant by "ratio"?

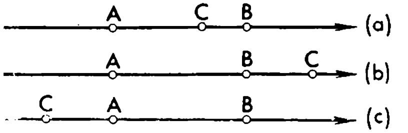


FIG. 4

All these questions will be answered later; for the present let us consider a directed line segment AB and a point C on it (Fig. 4*a*). We assume (also for the present) that A, B and C are different points, all of them. The ratio in which point C divides the segment AB is taken to be $\frac{AC}{CB}$. We denote it by the Greek letter λ:

$$\lambda = \frac{AC}{CB}. \tag{3.1}$$

The order of the letters in formula (3.1) should be noted, for all of them play different roles, viz:

A — is the initial point of the segment,
B — is the end point of the segment,
C — is the point of division.

The ratio in which the point divides the segment is constructed as follows:

the numerator—from the initial point to the point of division, the denominator—from the point of division to the end point.

In Fig. 4*a*, for example, point C divides the line segment AB in the ratio $\lambda = 2$.

Note that the definition given does not at all require that the point of division should lie inside the segment. In Fig. 4*b* point C lies outside the segment AB on the side of the end point. There is nothing to hinder us in the calculation of λ by formula (3.1). In this figure $AC > 0$, $CB < 0$, $\lambda = -3$. However, it is an unusual expression to say "point C divides the segment AB in the ratio $\lambda = -3$". We are accustomed to take the phrase "the point divides the segment" to mean that the point separates it in two parts, and in Fig. 4*b* the point C is outside the segment. However, if the reader is

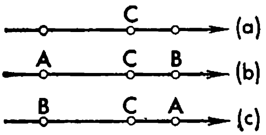


FIG. 5

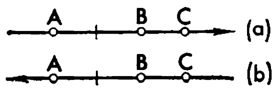


FIG. 6

a future mathematician, he should not be afraid of such difficulties. It is a constant occurrence in mathematics that notions and theorems get generalized, and the terminology remains unchanged, so that old terms and formulations are to be understood in a broader sense.

We say, by convention, that any point which lies inside a segment divides it *internally*, and an outside point—*externally*. The ratio λ is in all cases determined by formula (3.1). In Fig. 4*c*, for example, point C divides the segment AB externally in the ratio $\lambda = -\frac{1}{3}$.

We have thus made completely clear, what is to be understood by the ratio λ *for a directed line segment*. Let us now pose two questions:

(1) Is the fact of the segment being orientated of any consequence?

(2) Is the fact of the straight line on which the segment lies being orientated of any consequence?

Fig. 5*a* shows a non-orientated segment. In what ratio is it divided by point C? This question cannot be answered. In Figures 5*b* and 5*c* the segment has different orientations.

And the result? In Fig. 5*b* point C divides segment AB in the ratio $\lambda = 2$ and in Fig. 5*c* in the ratio $\lambda = \frac{1}{2}$.

So the first of our questions is to be answered in the affirmative. *The problem of dividing a line segment in a given ratio has no meaning if applied to a non-orientated segment.*

In order to answer the second question let us inspect Figures 6*a* and 6*b*. They differ only in the direction of the axis. It is obvious that *if the direction of the axis on which points A, B, C are lying is changed then all of the directed segments on this axis will only change their signs and consequently the ratio λ will remain unchanged.*

In Fig. 6*a* $AC = 3$, $CB = -1$, $\lambda = -3$.
In Fig. 6*b* $AC = -3$, $CB = 1$, $\lambda = -3$.

It follows that our second question is to be answered in the negative. In determining λ the orientation of the straight line on which the line segment lies is of no interest. Fig. 7 differs from Fig. 6 only in the segment AB being situated on a non-orientated straight line. This does not prevent us from finding that $\lambda = -3$.

FIG. 7

We cannot ascribe signs to line segments, which lie on a non-orientated straight line, but we can ascribe a sign to the ratio of the segments[1].

One need not know the sign of each individual segment in order to determine the sign of the ratio. The only relevant fact is whether the two segments have the same or opposite directions.

The problem of dividing a segment in a given ratio relates only to orientated segments, which lie on a non-orientated straight line.

However, if the straight line is not orientated how can we determine λ? Formula (3.1) does not apply in this case as it requires the segments to have signs.

If A, B, C are three points on a straight line then the ratio in which point C divides segment AB is the number λ, whose absolute value is equal to the ratio of the length of segment AC to that of CB,

$$|\lambda| = \frac{\overline{AC}}{\overline{CB}},$$

[1] We remind the reader that the segments are orientated.

the ratio being positive (negative) if point C is inside (outside) segment AB.

This definition is important since it shows that it is possible to determine λ on a non-orientated straight line.

However in solving any problem it is more convenient to apply a different method, viz. to give an orientation to the straight line, since we know that the value of λ is independent of its orientation. Besides, for an orientated straight line λ is completely determined (as to its absolute value and sign) by formula (3.1). This method is convenient for it allows to make use of the properties of directed line segments independently of the particular features of the drawing.

It remains to arrange for the cases when point C coincides with A or B. In the first case according to formula (3.1) we shall take $\lambda = 0$. In the second case formula (3.1) has no meaning (the denominator on the right-hand side becomes zero). In this case by convention $\lambda = \infty$, but this should be considered to be just a shorthand notation for the following fact: "the point of division coincides with the end of the segment". The symbol ∞ must not be treated as a number. No sign is ascribed to it (not always in mathematics, but in the problem considered)—it is neither $+\infty$ nor $-\infty$, it is simply ∞. There are reasons for this convention, which will be discussed slightly in Chapter II.

As the symbol $\frac{AC}{CB}$ is somewhat unwieldy, another one, which is simpler, is used, viz. (ABC):

$$\lambda = (ABC) \tag{3.2}$$

and a simpler term too: *a simple ratio of three points* (collinear, i.e., on a straight line). In the symbol of a simple ratio the initial point of the segment is given the first place, the end point of the segment—the second, and the point of division the third.

It has been made completely clear what the simple ratio λ is. We have now to formulate the problem of dividing a line segment in a given ratio. Here is the formulation:

Given a line segment AB and the number λ. It is required to find the point C, which divides the segment AB in the ratio λ.

Note. In all problems we shall assume that the segment given is not degenerated, i.e. take points A and B to be different points; λ can have any real value, i.e. $-\infty < \lambda < \infty$.

4. Solution of the problem. The formulation of a problem does not necessarily mean that it can be solved. And if it can be solved we do not know whether there is only one solution.

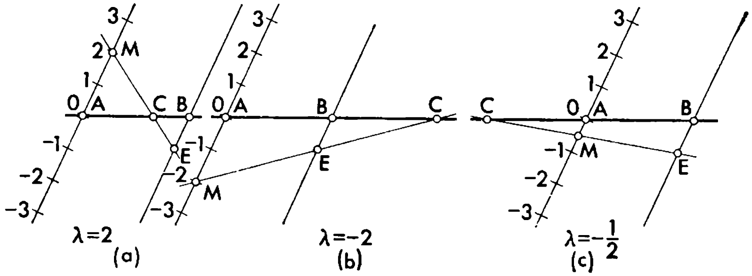


FIG. 8

It will be demonstrated first of all, that the problem cannot have more than one solution for any λ. Assume that there are two points C and C', which divide the segment AB in the same ratio

$$\frac{AC}{CB} = \frac{AC'}{C'B},$$

or breaking the segments in the numerators in two parts by a point B, we get

$$\frac{AB+BC}{CB} = \frac{AB+BC'}{C'B},$$

$$\frac{AB}{CB} - 1 = \frac{AB}{C'B} - 1,$$

$$\frac{AB}{CB} = \frac{AB}{C'B},$$

$$BC = BC'.$$

Hence, according to (2.6), point C' coincides with point C. It follows that if the problem has a solution for the given λ, this solution is the only one. Whether the problem has a solution in all cases is most conveniently determined in the process of solving.

Let us draw two parallel straight lines through points A and B (Fig. 8). Let us mark off a uniform scale on the first straight line, taking A as the initial point. Let us mark off one scale unit on the second straight line (i.e. $BE = A1$) in the opposite direction. Everything is now ready for solving the problem. We now find on the numerical axis point M which corresponds to the given value of λ and join M to E. The point of intersection of ME and AB is the required point C.

Indeed, AMC and BEC are similar triangles. Hence,

$$\frac{\overline{AC}}{\overline{CB}} = \frac{\overline{AM}}{\overline{BE}}.$$

Note: only the lengths of the segments enter into this proportion since elementary geometry and in particular the theory of similar triangles considers segments which have no signs. Since $\overline{BE} = 1$ we can rewrite the above proportion as follows:

$$\frac{\overline{AC}}{\overline{CB}} = |\lambda|.$$

The same argument holds for all the three cases shown in Fig. 8.

It remains now to prove that the signs of $\frac{AC}{CB}$ and λ coincide. Clearly if $\lambda > 0$ point C will be inside the segment and if $\lambda < 0$—outside the segment. Thus,

$$\frac{AC}{CB} = \lambda.$$

This construction does not result in obtaining point C only in the case, when the straight lines ME and AB prove parallel, which takes place with $\lambda = -1$. It follows that for $\lambda = -1$ either the above construction does not allow to find a solution, or there is no solution. It can be ascertained easily that there is no solution. Indeed, what does it mean to divide the segment AB in the ratio $\lambda = -1$? It means we have to find a point C, such that (1) is equally distant from points A and B (since $|\lambda| = 1$), and (2) is outside the segment AB (since $\lambda < 0$). No such point exists, since any point which lies outside the segment is nearer to one of its ends than to the other.

Note that for $\lambda = 0$ point C coincides with A and for $\lambda = \infty$ with B.

The problem of dividing a line segment in a given ratio has one and only one solution for any λ except $\lambda = -1$. With $\lambda = -1$ the problem has no solution.

To any value of λ (except $\lambda = -1$) there corresponds a definite point of the straight line AB and, conversely, for every point of the straight line AB there is a corresponding definite value of λ. The study of this correspondence is of some interest, i.e. it is interesting to represent clearly how

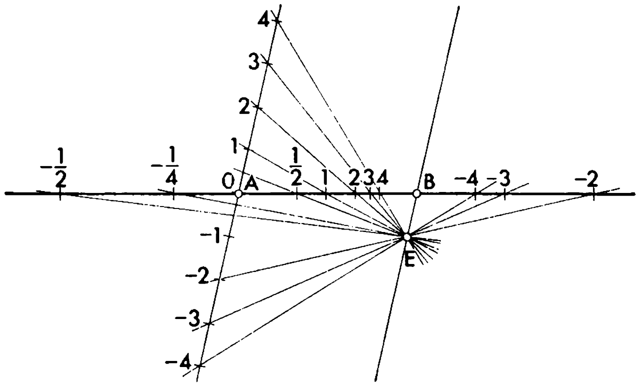


FIG. 9

the different values of λ are distributed over the straight line AB. This can be done geometrically or in analytical form.

The geometrical representation is based on the construction shown in Fig. 8. Draw through point E many straight lines and transfer along each of the lines the corresponding mark from the straight line $A1$ to the straight line AB (Fig. 9). The negative values less than unity in absolute value will fall outside the segment on the side of the initial point, and those greater than unity in absolute value—on the side of the end point of the segment.

Let us now state the analytical method. We introduce coordinates on the straight line AB. We take point A to be the origin and the direction of the axis to be from A towards B (though this is not a necessary condition). Point B will now have the coordinate a ($a > 0$), where a is the length of the line segment AB. Now take any point on the axis

$C\ (x)$. Then

$$\frac{AC}{CB} = \lambda,$$

or by formula (2.5)

$$\frac{x-0}{a-x} = \lambda.$$

Hence $x = \frac{\lambda a}{1+\lambda}$. Thus we have two formulas which give λ in terms of x and x in terms of λ:

$$\left. \begin{aligned} \lambda &= \frac{x}{a-x}, \\ x &= \frac{\lambda a}{1+\lambda}. \end{aligned} \right\} \qquad (4.1)$$

Choosing different points on the axis we can determine λ by the former formula in (4.1). Conversely assuming a value of λ the latter formula makes it possible to find point x and plot it in the drawing. The direction of the axis does not tell on the result.

5. Mechanical interpretation of the problem. Let us place at points A and B masses m_1 and m_2, respectively, and find the centre of gravity of these two material points. It lies on the line segment AB and divides it in two parts inversely proportional to the adjacent masses, i.e.

$$\frac{AC}{CB} = \frac{m_2}{m_1}$$

(C denotes the required centre of gravity). Thus the problem of dividing a line segment AB in the ratio $\lambda = \frac{3}{2}$ may be interpreted as follows: place mass 2 at point A and mass 3 at point B, then the centre of gravity is the required point.

This interpretation has a shortcoming: it can be applied only for $\lambda > 0$. In order to make it serve for the case $\lambda < 0$ we should have to introduce negative masses.

6. Invariant property of a simple ratio with respect to parallel projection. The term "invariant property" means that the ratio does not change. The title of this section expresses the following property:

If three points on a straight line (collinear) are projected by parallel lines (projectors) onto another straight line then their simple ratio remains unchanged.

The meaning of the expression "projected by parallel lines" is clearly explained by Fig. 10.

Projectors a, b, c are drawn through points A, B, C. The points of intersection of these lines with the straight line onto which we project the points, i.e. A', B', C', are called parallel projections of the points A, B, C.

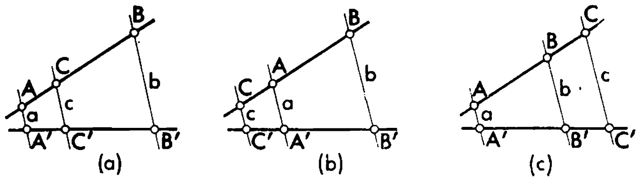


FIG. 10

Proof. By the known theorem on the proportionality of line segments cut out by parallel straight lines on the sides of an angle we have

$$\frac{\overline{A'C'}}{\overline{C'B'}} = \frac{\overline{AC}}{\overline{CB}}$$

or

$$\left|\frac{A'C'}{C'B'}\right| = \left|\frac{AC}{CB}\right|.$$

It remains to be shown that the simple ratios $\frac{A'C'}{C'B'}$ and $\frac{AC}{CB}$ are of the same sign.

This is clearly the case: if point C lies between A and B, then point C' lies between A' and B' (Fig. 10a), and the two ratios are positive. If point C is outside the segment AB, then point C' also is outside the segment $A'B'$ (Fig. 10b and 10c) and the two ratios are negative. Thus, it has been proved that

$$\frac{A'C'}{C'B'} = \frac{AC}{CB}, \qquad (6.1)$$

or $(A'B'C') = (ABC)$.

This property can be considered from another viewpoint:

Three parallel straight lines a, b, c cut out on any straight line (if it is not parallel to them) the same simple ratio.

Thus in Fig. 11

$$(A_1B_1C_1) = (A_2B_2C_2) = (A_3B_3C_3) = \ldots$$

Since the simple ratio is independent of the cutting line, it belongs to the three lines a, b, c.

We can now take a further step. So far we had to do only with the simple ratio of three points on a straight line, now we shall introduce the notion of the simple ratio of three parallel straight lines.

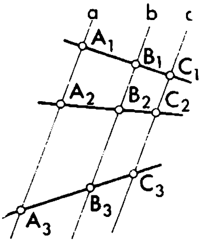


FIG. 11

A simple ratio of an ordered triad of parallel straight lines is the simple ratio of three points cut out by these three lines on any cutting straight line.

7. Permutation of elements in a simple ratio. Fig. 12 shows three points on a straight line. What is their simple ratio? This question cannot be answered for the points are not ordered. They can be ordered in various ways. Therefore to this non-ordered triad of points there correspond several different values of λ. How many?

There are six possible ways of ordering three points: (ABC), (BAC), (ACB), (CAB), (BCA), (CBA). Let us orientate in some way the straight line, on which our points are lying, and denote the simple ratio (ABC) by λ:

FIG. 12

$$\lambda = (ABC) = \frac{AC}{CB}. \qquad (7.1)$$

In the following derivation use will be made of the properties of directed segments, (2.1) and (2.3). Every time we come across segment AB or BA we shall divide it by point C.

Let us calculate the remaining five ratios:

$$(BAC) = \frac{BC}{CA} = \frac{-CB}{-AC} = \frac{1}{\lambda}.$$

Note: *if we interchange the initial point and the end point of a segment the simple ratio will be changed to its inverse.*

Further

$$(ACB) = \frac{AB}{BC} = \frac{AC+CB}{-CB} = -\frac{AC}{CB} - 1 = -(1+\lambda).$$

The next simple ratio (CAB) need not be calculated by the above method; we can apply the rule of permutation of the initial and end points that has just been stated:

$$(CAB) = -\frac{1}{1+\lambda}.$$

Further

$$(BCA) = \frac{BA}{AC} = \frac{BC+CA}{AC} = \frac{-CB-AC}{AC} =$$

$$= \frac{-1-\frac{AC}{CB}}{\frac{AC}{CB}} = -\frac{1+\lambda}{\lambda}.$$

Interchanging once more the initial and end points we obtain

$$(CBA) = -\frac{\lambda}{1+\lambda}.$$

Let us tabulate the results obtained. It is undesirable to assign definite letters to the points, since in different cases other letter notations might occur. Only the places occupied by the *elements* (possibly not points but straight lines) of a simple ratio are important. Therefore we shall substitute numbers 1, 2, 3 for the letters A, B, C.

$$\left.\begin{array}{ll} (123) = \lambda, & (312) = -\frac{1}{1+\lambda}, \\ (213) = \frac{1}{\lambda}, & (231) = -\frac{1+\lambda}{\lambda}, \\ (132) = -(1+\lambda), & (321) = -\frac{\lambda}{1+\lambda}. \end{array}\right\} \qquad (7.2)$$

Thus, one and the same non-ordered triad generates several simple ratios depending on the method of ordering. For example, the following simple ratios correspond to the triad shown in Fig. 12: 2, $\frac{1}{2}$, -3, $-\frac{1}{3}$, $-\frac{3}{2}$ and $-\frac{2}{3}$.

How many different simple ratios correspond to a non-ordered triad? Generally speaking, six, as the summary (7.2) shows. Why do we say "generally speaking" in this formulation? Because the values given in table (7.2) are not always different. With some special arrangements of the points some of the values may coincide. Let us find these cases.

The problem is very interesting. It seems at first sight that there are very many such triads, but it turns out that there is only one.

To solve this problem we have to take any pairs of the values in table (7.2), equate them, and find λ. It should be noted, however, that we denoted by λ not one definite simplė ratio but any one of the six. It is sufficient to equate λ to the remaining expressions. This reduces the number of possible variants to five. Let us assume that all the three points are different. Degenerated triads (whose elements coincide) are of no interest: obviously if two points coincide their permutation does not change the simple ratio. This means that we must consider as unacceptable the values $\lambda = 0$ and $\lambda = \infty$.

Consider now the five variants:

(1) $\lambda = \frac{1}{\lambda}$, $\lambda^2 = 1$, $\lambda = \pm 1$. Since $\lambda = -1$ is impossible, only the solution $\lambda_1 = 1$ holds.

(2) $\lambda = -(1+\lambda)$. Hence $\lambda_2 = -1/2$.

(3) $\lambda = -\frac{1}{1+\lambda}$; $\lambda^2 + \lambda + 1 = 0$. The roots are imaginary.

(4) $\lambda = -\frac{1+\lambda}{\lambda}$. The same as above.

(5) $\lambda = -\frac{\lambda}{1+\lambda}$. Rejecting the unsuitable solution $\lambda = 0$, we have $\lambda_3 = -2$.

We have obtained three values: $\lambda_1 = 1$; $\lambda_2 = -1/2$; $\lambda_3 = -2$.

Recall now, that we seek a non-ordered triad to which there correspond six values of λ and not one only.

Using the three values of λ, we now construct the six ratios into which these values enter (see the table on p. 23).

The three rows of this table coincide except for the order of the values. Consequently the three found values of λ should be considered as one solution. They correspond to a triad of points one of which is the midpoint of the segment bounded by the two others.

There are generally speaking six different values of the simple ratio corresponding to a non-ordered not degenerated triad. There is one exception, the triad, whose third point is in the middle of the segment bounded by the two other points at its ends. There are only three different values of a simple ratio corresponding to such a triad.

λ	$\frac{1}{\lambda}$	$-(1+\lambda)$	$-\frac{1}{1+\lambda}$	$-\frac{1+\lambda}{\lambda}$	$-\frac{\lambda}{1+\lambda}$
1	1	−2	−1/2	−2	−1/2
−1/2	−2	−1/2	−2	1	1
−2	−1/2	1	1	−1/2	−2

Clearly the interchanging of the extreme points of the segment in such a triad does not tell and cannot change the value of the simple ratio.

8. Group property of a simple ratio. The notion of a group is one of the fundamental notions in mathematics. It cannot be mentioned casually or spoken of glibly. We are going to speak of the group property of a simple ratio without any connection with the general notion of a group. The reader of this booklet should for the present take the contents of this section as an isolated fact, as an interesting property of a simple ratio.

Later on, having become familiar with the theory of groups[1], the reader will see that this property relates to profound ideas. This also is an example of studying in depth!

The group property of a simple ratio can be stated as follows:

If one of the following six expressions (for λ*)*

$$\left.\begin{array}{l} a_1=\lambda,\quad a_2=\dfrac{1}{\lambda},\quad a_3=-(1+\lambda),\\ a_4=-\dfrac{1}{1+\lambda},\quad a_5=-\dfrac{1+\lambda}{\lambda},\quad a_6=-\dfrac{\lambda}{1+\lambda}. \end{array}\right\} \qquad (8.1)$$

is substituted for any other, the result will be again one of the six.

This fact may be considered from another point of view. Each a_i is a function of λ:

$$a_i = f_i(\lambda) \quad (i = 1, 2, \ldots, 6).$$

[1] The reader can start with the books (1) *Introduction to the Theory of Groups*, by P. S. Aleksandrov, 2nd ed., Moscow, 1951, (2) *Groups and Their Graphs*, by I. Grossman and W. Magnus, New York, 1964.

Substituting one of these functions for the argument we obtain the following expression:

$$f_i\,[f_j\,(\lambda)]\ (i = 1, 2, \ldots, 6;\ j = 1, 2, \ldots, 6) \qquad \text{(a)}$$

(the case where $i = j$ is not excluded). The function (a) proves to be not a new one, it is one of the same six:

$$f_i\,[f_j\,(\lambda)] = f_k\,(\lambda). \qquad (8.2)$$

Thus the operation under consideration (the substitution of one of the functions $f_i\,(\lambda)$ for λ) results in nothing new, i.e. we do not get beyond the original six functions.

How can this be proved? We can try all of the 36 combinations of i and j in formula (8.2.). Take, for example, $i = 3$, $j = 6$. This means that we take

$$a_3 = -(1 + \lambda)$$

and substitute $a_6 = -\frac{\lambda}{1+\lambda}$ for λ

$$-(1 + a_6) = -\left(1 - \frac{\lambda}{1+\lambda}\right) = -\frac{1}{1+\lambda} = a_4.$$

It follows that

$$f_3\,[f_6(\lambda)] = f_4\,(\lambda).$$

However this method of proving will not satisfy an inquisitive reader. If 36 checks confirm our assumption, we cannot think that we have to do with 36 accidental coincidences. There must exist some simple reason for this fact. We are going to point out this reason and our proof will become much more convincing.

We denoted by λ *any* of the six simple ratios (ABC), (BAC), ... In table (7.2) we find, for instance,

$$(123) = \lambda,\ (213) = \frac{1}{\lambda}.$$

Paying no attention to the notations, this means: if the inverse value is substituted for the value of a simple ratio this corresponds to the interchanging of the first two elements of the triad. This statement applies to any of the six ratios (since it does not matter which of the three points is denoted by the number 1, etc.). It follows that by substituting the inverse value for $-\frac{1+\lambda}{\lambda}$ (i.e. substitute in $\frac{1}{\lambda}$ the value $-\frac{1+\lambda}{\lambda}$ for λ) we obtain a simple ratio whose two first elements have been interchanged; this ratio is of course tabulated in (7.2). Q.E.D.

Let us turn again to formula (8.2). We have found that for $i = 3$, $j = 6$ we get $k = 4$. Let us solve the problem comprehensively, i.e. find k for any pair i, j.

First, let us remember what is understood in arithmetic and algebra by a *binary operation*, i.e. an operation involving two components (there are operations with one component, too, for instance, extracting a square root). Assume that we are given a certain set M. *A binary operation is one that makes one and only one element of set M correspond to any ordered pair* (a, b) *of elements of this set*[1].

Example 1. Let M be the set of natural numbers 1, 2, 3, ... The operation of *addition* of each pair of numbers makes one and only one element of the set—their sum—correspond to the pair. The sign of the operation is often written between the components of the pair: $2 + 3 = 5$.

Example 2. Consider the operation of *multiplication* of elements of the same set. This is a *different* operation. It makes not 5 but 6 correspond to the same pair (2, 3): $2 \cdot 3 = 6$.

The two operations have the same property, they are *commutative* ($a + b = b + a$, $a \cdot b = b \cdot a$) and therefore it does not matter whether the pair is ordered or not. However if we consider the operation $a^b = c$ we see that its components are not equivalent in their roles.

Consider now a set M whose elements are the functions (8.1). We define for this set an operation which we shall call "multiplication" and denote by the sign $\odot$ (we use quotation marks and a circle in order to avoid confusion with the operation of real multiplication).

To "multiply" a_i *by* a_j *means to substitute* a_j *for* λ *in* a_i. *In symbolic form*

$$a_i \odot a_j = f_i\,[f_j\,(\lambda)] = a_k. \tag{8.3}$$

It has been shown above that $f_3\,[f_6\,(\lambda)] = f_4\,(\lambda)$.

This can now be formulated as follows: if a_3 is "multiplied" by a_6 we obtain a_4 or

$$a_3 \odot a_6 = a_4.$$

---

[1] This is not absolutely necessary, but we shall limit ourselves to this case.

The reader should find all of the 36 "products" $a_i \odot a_j$. The results are tabulated below.

First factor	Second factor					
	a_1	a_2	a_3	a_4	a_5	a_6
a_1	a_1	a_2	a_3	a_4	a_5	a_6
a_2	a_2	a_1	a_4	a_3	a_6	a_5
a_3	a_3	a_5	a_1	a_6	a_2	a_4
a_4	a_4	a_6	a_2	a_5	a_1	a_3
a_5	a_5	a_3	a_6	a_1	a_4	a_2
a_6	a_6	a_4	a_5	a_2	a_3	a_1

(8.4)

Table (8.4) may be called a "multiplication" table[1]. Let us take a close look at it and make some observations.

1. "Multiplication" is not commutative. We have, for example, $a_2 \odot a_3 = a_4$, but $a_3 \odot a_2 = a_5$. Therefore when saying "multiply by a_i" we must add "on the right" or "on the left". For example, "multiply a_2 on the right by a_3" means $a_2 \odot a_3 = a_4$ and "multiply a_2 on the left by a_3" means $a_3 \odot a_2 = a_5$.

2. In "multiplying" the element a_1 plays the same role as 1 in ordinary multiplication. Multiplying any number by unity does not change this number.

$$a \cdot 1 = a.$$

As can be seen from table (8.4) "multiplication" (both on the right and on the left) of any element by a_1 does not change the element:

$$a_i \odot a_1 = a_1 \odot a_i = a_i \quad (i = 1, 2, \ldots, 6).$$

[1] In the theory of groups it is known as Cayley square.

This is why the element a_1 is called "unity".

3. "Multiplication" is *associative*

$$(a_i \odot a_j) \odot a_k = a_i \odot (a_j \odot a_k). \quad (8.5)$$

For example, if we *first* "multiply" $a_2 \odot a_3$ and *then* "multiply" (on the right) the result by a_4 we obtain

$$(a_2 \odot a_3) \odot a_4 = a_4 \odot a_4 = a_5.$$

But if we *first* "multiply" a_3 by a_4, $a_3 \odot a_4$, we obtain

$$a_2 \odot (a_3 \odot a_4) = a_2 \odot a_6 = a_5.$$

The results are identical.

We can in this way check up all the combinations and find that the law (8.5) holds.

In writing down the product of three elements and more the use of parentheses is superfluous due to the associative law.

We can simply write

$$a_i \odot a_j \odot a_k$$

meaning any of the sides of the equality (8.5).

We can define the operation of "division". But, enough! We must stop. We leave to the reader to think it over by himself.

A set of elements for which an operation with certain properties, which we shall not enumerate here, has been defined is called a *group*.

The elements (8.1) with the operation, defined by the formula (8.3), form a group.

We just peeped into the theory of groups through a chink, we wish the reader of this booklet to enter into the theory later on through a wide open door.

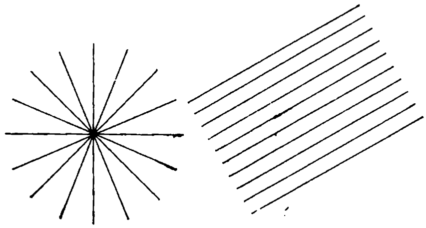
9. Ideal points (points at infinity). In this section we shall extend the notion of a point, otherwise our further progress would suffer. The difficulties that could be come across will be discussed in Section 10. Let us by convention ascribe a point in common to parallel lines and call it *an ideal point*[1].

Thus we shall further use the expression, that *two parallel lines intersect at an ideal point.*

---

[1] It is also called *the point at infinity* (the point of intersection of parallel lines). This phrase is less adequate in our case since we shall never have to do with the distance to this point.

The reader naturally feels the need to consider this point just as any other. Where is it? However this point is not a point like any other—it is an *ideal point*. It can also be made palpable but not in the same way as *an ordinary point*; we shall now attempt to overcome the natural resistance of our organism to the introduction of new concepts, which are incompatible with our habitual desire of perceptibility.



(a) (b)

FIG. 13

The ideal point is the one which parallel straight lines have in common. Two parallel lines have the same *direction*. Thus the introduction of ideal points is not a radical revolution but only a modest renaming: from now on the term "direction of a straight line" will be replaced by a new one—"ideal point".

Fig. 13*a* shows many straight lines which pass through one and the same point. This set of lines is called *a central pencil* and the point they have in common *the pencil centre.* Clearly, the pencil is completely described by giving its centre, and vice versa. Fig. 13*b* shows a pencil of *parallel lines*, i.e. a set of parallel straight lines lying in one plane. If instead of points we think of pencils of lines the difference between ordinary and ideal points is partly removed:

an ordinary point is a central pencil,
an ideal point is a parallel pencil.

We caution the reader not to ask where the ideal point is—on the right or on the left. This would be an attempt to approach new notions using the old intuition. To learn how to handle ideal points one must develop a new intuition.

There is one and only one ideal point on every straight line and the notions "on the right", "on the left", "over", "under", etc., are not applicable to it. Fig. 14 shows a straight line a and a central pencil S. Let us determine the correspondence between the points of the straight line and the lines of the pencil: point M corresponds to the straight line m, and vice versa (see the figure). Can we say that it is a one-to-one correspondence, i.e. *that to every point on the line a there is a corresponding line of the pencil S and vice versa, to every line of the pencil S there is a corresponding point on the line a*? Prior to the introduction of ideal points

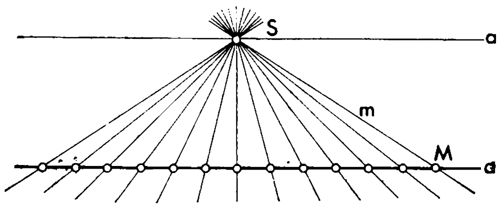


FIG. 14

the second part of the above statement would be wrong: there is one superfluous straight line a' in the pencil S; this line is parallel to a. No point on the straight line a corresponds to the line a'. However after the introduction of ideal points our statement holds true. There is now on the straight line a a point which corresponds to the straight line a', viz. the ideal point. It is neither on the right nor on the left. If the straight line m is rotated about point S counterclockwise, then point M will move along the straight line a to the right, and if m is rotated clockwise, then point M will move to the left. In both cases at a certain moment the straight line m will coincide with a'. At this moment point M will become ideal.

Clearly, there is an infinite set of ideal points in a plane. Let us denote this set by u. We shall take it, by convention, to form an ideal straight line. This is a natural assumption for two reasons.

First of all each ordinary straight line has one ideal point, i.e. a point in common with the set u. It is natural to take u to be a straight line.

Secondly, let us consider two parallel planes α and α' (Fig. 15). Each parallel pencil in plane α corresponds to a parallel pencil of the same direction in plane α'. In other words, each ideal point in plane α belongs also to plane α'

and vice versa. This means that the set of ideal points of the parallel planes is common. This is one more argument for calling this set a straight line.

Thus, *there exists one and only one ideal straight line in a plane. Each ordinary straight line has one and only one ideal point, and the ideal straight line consists of only ideal points.*

If two ideal points are found on a straight line, then this line is an ideal one.

The set comprising all ideal points of space is called *an ideal plane.* We shall not have to do with it, since our booklet is devoted to plane geometry.

We have extended the set of points on the plane by introducing new points. Can it be assumed that these new points are equivalent to the old ones, i.e. that they differ in nothing from the ordinary points? No. Ideal points are equivalent to the ordinary ones only in *several respects.* Let us point out in what respects.

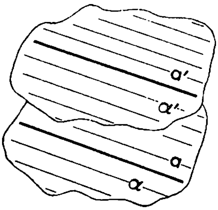


FIG. 15

In positional problems, i.e. in problems relating to the coincidence of a point with a point in the straight line there is no difference between ordinary and ideal points. Indeed, the positional properties in a plane follow from two axioms.

1. *Two different points determine one and only one straight line* (passing through them, it is implied).

2. *Two different straight lines determine one and only one point* (belonging to both, it is implied).

The first axiom is illustrated in Fig. 16. Each point is given by a pencil. To draw a straight line through two points means to find a straight line which the two pencils have in common. In Fig. 16*a* the two points are ordinary ones. This is the "old" well-known case. In Fig. 16*b* one of the points is ordinary and the other ideal. Clearly, they determine one and only one straight line in this case too. In Fig. 16*c* the two points are both ideal. In this case there is also one and only one straight line on which these points lie: it is an ideal line.

For checking up the second axiom we shall consider three cases.

(1) The two straight lines are ordinary non-parallel lines. They intersect at an ordinary point.

(2) The two straight lines are ordinary parallel lines. They have one point in common—an ideal one.

(3) One of the lines is ordinary, the other ideal; the only point which they have in common is the ideal point of the first line.

In metrical problems, i.e. in problems relating to the measurement of line segments and angles ideal points are not

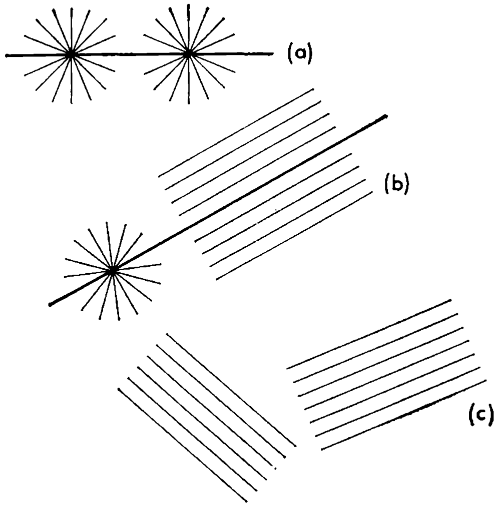


FIG. 16

equivalent to ordinary ones. We cannot speak of a distance between two ideal points (but we can speak of an angle!). One and only one perpendicular can be dropped from an ordinary point onto an ordinary straight line, but either none or an infinite number of perpendiculars can be dropped from an ideal point, etc.

The axiom on parallel lines does not hold either for the case when either the straight line or the point outside it are ideal.

Let us now turn back to the subject of this booklet. If two points A and B are ordinary and U is the ideal point of the straight line AB, then what is the simple ratio (ABU)? This certainly is a matter of convention, but the convention must be reached in the most natural way.

If all the three points are ordinary, then

$$\lambda = (ABC) = \frac{AC}{CB} = \frac{AB+BC}{CB} = \frac{AB}{CB} - 1;$$

if point C *tends to* U (this means it moves away along the line indefinitely), then $\lim\limits_{C \to U} \frac{AB}{CB} = 0$. Therefore if C coincides with U it is natural to ascribe λ the limit value:

$$\lambda = (ABU) = -1.$$

So far the value $\lambda = -1$ was considered impossible. Now it proved just suitable for the ideal point.

From now on it is possible to divide the segment AB in *any* ratio.

Let us further consider the cases when the initial point or the end point of a line segment is an ideal point.

If $A \to U$ then the numerator of the expression $\lambda = \frac{AC}{CB}$ increases indefinitely and the denominator is a constant value, and vice versa if $B \to U$. Therefore it is natural to take

$$\left.\begin{aligned} (UBC) &= \infty, \\ (AUC) &= 0, \\ (ABU) &= -1. \end{aligned}\right\} \qquad (9.1)$$

Not all of the properties of a simple ratio, discussed above, hold also for ideal elements.

10. Separation of points on a straight line. The number of points on a straight line is infinite. We have added one

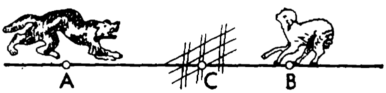


FIG. 17

more point to it—the ideal one. Does it really matter?

Oh, yes! The addition of the ideal point produces an essential change in the properties of a straight line. In particular, the notion of "betweenness" loses its meaning after the introduction of the ideal point.

So far we assumed that of three points of a straight line two are always *extreme* points and the third *interstitial*, i.e. lying *between* the extreme ones. We also say that one of the points *separates* the other two. If there is a wolf at point

A, and a sheep at point B (Fig. 17) then we can provide for the safety of the sheep by placing at point C an insurmountable barrier (it is assumed that the wolf can move only along the straight line). In such a case the barrier separates the wolf from the sheep. These properties of a straight line do not hold on a circle. Among three points on a circle there is no definite interstitial point. If it is required to separate

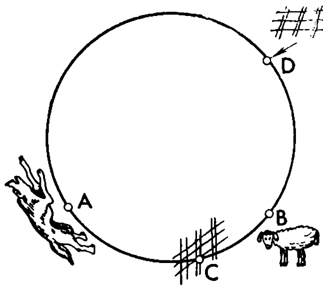


FIG. 18

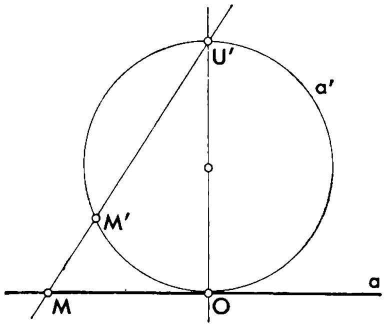


FIG. 19

the wolf from the sheep then one barrier is not enough. The barrier at point C (Fig. 18) will not prevent the wolf from reaching the sheep, moving clockwise. Two barriers are now necessary—at C and D, then only can the sheep feel safe.

Thus, *on a circle one point cannot separate a pair of points, and two—can.* Moreover, *a tetrad of points on a circle breaks up into two separate pairs in one and only one way.*

After the introduction of the ideal point this difference between the straight line and the circle vanishes. Now the straight line is a closed one.

Fig. 19 shows that between points on a circle and points on a straight line there is a correspondence which is called a *stereographic projection.* The straight line touches the circle a' at point O. The diametrically opposite point U' is the centre of projection. A point M' on the circle corresponds to each point M on the straight line, and vice versa a point M corresponds to each point M'. Until the introduction of the ideal point this mapping was not a one-to-one mapping: there was one superfluous point, U'. Now the ideal point U of the straight line a corresponds to point U'.

In a central pencil of lines a single straight line cannot separate two lines of the pencil either. Let a and b be two lines in the pencil (Fig. 20). Whatever the third straight line c, the line a or b can always be rotated in such a way that it does not pass through the position occupied by c. However two straight lines can separate a and b. Moreover, *a tetrad of straight lines in a central pencil breaks up into two separate pairs in one and only one way.*

Consider again Fig. 14. One can easily ascertain that the following statement holds true.

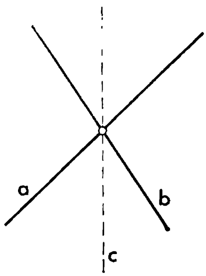


FIG. 20

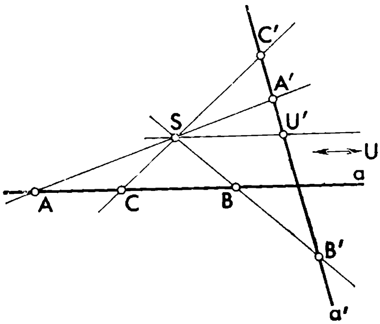


FIG. 21

Let a tetrad of points on a straight line be projected from a certain point by a tetrad of straight lines. In this case separate pairs of points are projected by separate pairs of straight lines.

Thus, the mutual separateness of two pairs of points remains with central projection. This is true also of the projection of one straight line on another one. The property of a triad of points of a straight line to break up into a pair of extreme points and one interstitial point (only with a straight line that has no ideal point!) remains only with parallel projection (see Fig. 11), but not with central projection. Fig. 21 shows three points A, B, C on a straight line a with point C lying *between* A and B. These points are projected from centre S onto the straight line a' and point C' *does not lie* between A' and B'.

Let the reader pause in looking at Fig. 21 and think over the question: is the line segment $A'B'$ the projection of the segment AB? The question as it stands is not quite clear, since there are two segments AB on the straight line a:

the internal one and the external one, i.e. the one that has an ideal point. It can be seen clearly in Fig. 21 that any internal point of the line segment AB (for instance, C) is projected onto an external point of the segment $A'B'$. If, however, we project the ideal point U of our straight line (the projecting ray is parallel to a) we obtain the point U' inside the segment $A'B'$. It follows that the internal segment $A'B'$ is the projection of the external segment AB.

These abstract reasonings are of practical value for the wolf A (Fig. 17). After the introduction of the ideal point the barrier C will not prevent the wolf from reaching the

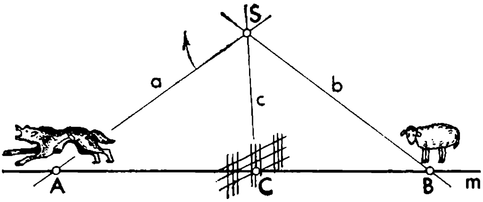


FIG. 22

sheep at point B, he will only have to move via the ideal point. Let us imagine that all the three points are projected from point S (Fig. 22). The straight line a rotates clockwise about point S. The wolf must run to the left in a way such that at any given moment he is at the point of intersection of line a with m. When a becomes parallel to m the wolf will be at the ideal point. With further rotation of line a the wolf will appear on the right and reach the sheep.

However the wolf must be a racer of the highest class. If the line a rotates about S at a uniform speed, then the speed of the wolf, when a nears the position parallel to m, must increase indefinitely. We should not feel any surprise at this. Speed is a metric notion related to the measuring of distances, and the reader was warned not to consider metric problems in connection with ideal points.

11. Ceva's theorem. The medians of a triangle pass through one and the same point. There is no converse to this theorem: from the fact that straight lines AL, BM and CN pass through the same point (Fig. 23), it does not follow that these lines are medians. The absence of the converse of the theorem means that its formulation comprises more conditions than are required: in order to conclude that the straight lines

AL, BM and CN pass through the same point it is not necessary to require them to be medians (or heights or bisectors); we can content ourselves with *weaker* conditions. It is interesting to find the *minimum* condition, i.e. such, that if it is not observed, the lines AL, BM and CN cannot have a point in common. Such a theorem will have a converse. This condition will be sufficient and necessary.

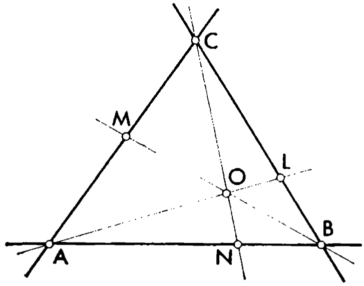


FIG. 23

This minimum condition was found by the Italian mathematician Giovanni Ceva (1647-1734). Before formulating the theorem let us give precise definitions of some terms.

We shall use the phrase "the side of a triangle" not for a line segment, but for the whole line of indefinite extent. Given a triangle ABC we shall select one point on each side (which does not coincide with any of the vertices of the triangle):

on side AB we take point N,
on side BC we take point L,
on side CA we take point M.

In order to determine the ratios in which these points divide the sides of the triangle we must order its vertices. Let us agree to go around the triangle in any of the two directions: either ABC or ACB. In the first case the pairs of vertices will be ordered as follows:

$$AB,\ BC,\ CA,$$

in the second the order will be reversed

$$BA,\ AC,\ CB.$$

For definiteness we shall choose one direction, the first, and denote

$$\left.\begin{aligned} \lambda &= (BCL), \\ \mu &= (CAM), \\ \nu &= (ABN). \end{aligned}\right\} \qquad (11.1)$$

Ceva's theorem says:

If the straight lines AL, BM and CN pass through one and the same point, then $\lambda\mu\nu = 1$.

Note. Points A, B, C are ordinary and points L, M, N and O can be of any kind (ordinary or ideal).

Proof. Assume first that all seven points are ordinary. Point O can be inside the triangle and outside it.

In order to make our proof more palpable we give two drawings, Figs. 24*a* and *b*. The following reasoning applies to both drawings.

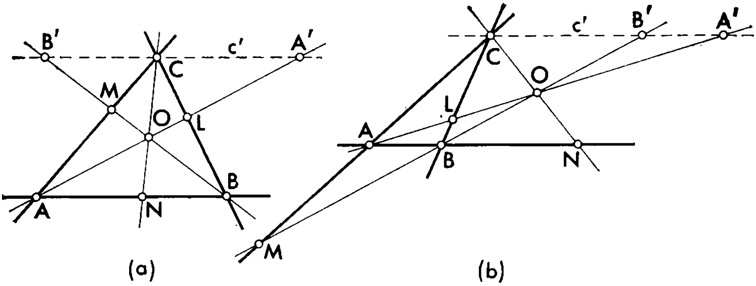


FIG. 24

Draw a straight line $c' \parallel AB$ through point C. Denote by A' and B' the points of intersection of AL and BM with c', respectively. Further in our proof we use proportions which follow from the similarity of triangles and, consequently, *regard all line segments and their ratios as positive.*

From similar triangles $OA'C$ and OAN we get

$$\frac{\overline{AN}}{\overline{CA'}} = \frac{\overline{ON}}{\overline{OC}}.$$

From similar triangles $OB'C$ and OBN we obtain

$$\frac{\overline{NB}}{\overline{CB'}} = \frac{\overline{ON}}{\overline{OC}}.$$

Equating the left-hand sides of the proportions we can write

$$\frac{\overline{AN}}{\overline{CA'}} = \frac{\overline{NB}}{\overline{CB'}},$$

or, permutating the terms of the proportion,

$$\frac{\overline{AN}}{\overline{NB}}=\frac{\overline{CA'}}{\overline{CB'}}. \tag{a}$$

From similar triangles ABL and $A'CL$ we have

$$\frac{\overline{BL}}{\overline{LC}}=\frac{\overline{AB}}{\overline{CA'}}. \tag{b}$$

From similar triangles ABM and $CB'M$ we obtain

$$\frac{\overline{CM}}{\overline{MA}}=\frac{\overline{B'C}}{\overline{AB}}. \tag{c}$$

Multiplying (a), (b) and (c) we obtain

$$\frac{\overline{AN}}{\overline{NB}}\cdot\frac{\overline{BL}}{\overline{LC}}\cdot\frac{\overline{CM}}{\overline{MA}}=1. \tag{d}$$

It remains to find out what will be the result of introducing into the left-hand side of (d) the ratios (11.1) with their signs. If point O is inside the triangle, as shown in Fig. 24*a*, then each of the points L, M and N is *between* two vertices of the triangle and, consequently, all of the three ratios (11.1) are positive. This means that in this case instead of (d) we can write

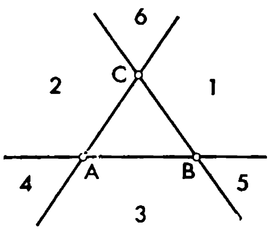


FIG. 25

$$\frac{AN}{NB}\cdot\frac{BL}{LC}\cdot\frac{CM}{MA}=1. \tag{11.2}$$

If point O is outside the triangle (but not on its side) it lies either inside an angle of the triangle (domains *1*, *2*, *3* in Fig. 25) or inside one of the corresponding vertical angles (domains *4*, *5*, *6* in Fig. 25). Let, for example, point O lie inside angle A or its corresponding vertical angle. In this case the straight line AO will intersect the side BC at point L, which is *between* B and C, and the lines BO and CO will intersect the sides CA and AB *outside* the line segments CA and AB. Thus, if point O is outside the triangle ABC, of the three ratios (11.1) one will be positive and two negative. This means that the product $\lambda\mu\nu$ in any case will be positive and proves that the relationship (11.2) holds true.

Let now one of the points of division, for instance L, be ideal, i.e. $AL \parallel BC$ (Fig. 26). It follows from similar triangles AON and BCN that

$$\frac{\overline{AN}}{\overline{NB}} = \frac{\overline{OA}}{\overline{BC}}.$$

From similar triangles OMA and BMC we have

$$\frac{\overline{MA}}{\overline{CM}} = \frac{\overline{OA}}{\overline{BC}}.$$

Equating the left-hand sides we obtain

$$\frac{\overline{AN}}{\overline{NB}} = \frac{\overline{MA}}{\overline{CM}}. \qquad \text{(e)}$$

Since point O must lie on the straight line AL which is parallel to BC, the point is in the hatched domain in Fig. 27.

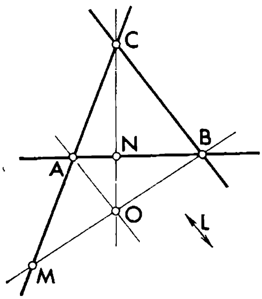


FIG. 26

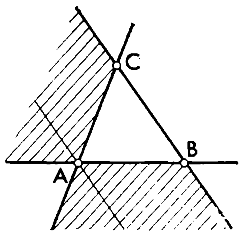


FIG. 27

It is easily ascertained that in this case either point N lies between A and B and point M—outside the line segment AC, or vice versa, i.e. the ratios $\frac{AN}{NB}$ and $\frac{MA}{CM}$ are of opposite signs. Therefore (e) can be rewritten as follows:

$$\frac{AN}{NB} = -\frac{MA}{CM},$$

or

$$\frac{AN}{NB} = -\frac{1}{\frac{CM}{MA}},$$

i.e.

$$\nu = -\frac{1}{\mu}.$$

Thus, $\mu\nu = -1$. Point L is an ideal one, i.e. $\lambda = -1$. Thus,

$$\lambda\mu\nu = 1.$$

Consider further the case when two points of division, for example L and M, are ideal. In this case the figure $ACBO$ is a parallelogram (Fig. 28), and point N is the midpoint of the side AB. In this case $\lambda = -1$, $\mu = -1$, $\nu = 1$ and, consequently, $\lambda\mu\nu = 1$.

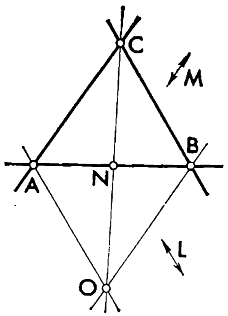


FIG. 28

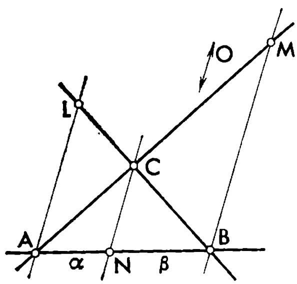


FIG. 29

All three points L, M, N cannot be ideal (if it is given that the lines AL, BM and CN have a point in common). A possible case is that O is an ideal point, i.e. the straight lines AL, BM and CN are parallel (Fig. 29). It is easily ascertained that in this case one of the ratios is positive and the two remaining negative. In Fig. 29 $\lambda < 0$, $\mu < 0$, $\nu > 0$ Let us denote the line segments AN and NB by α and β, respectively. Then

$$\nu = \frac{AN}{NB} = \frac{\alpha}{\beta},$$

$$\lambda = \frac{BL}{LC} = -\frac{\alpha+\beta}{\alpha},$$

$$\mu = \frac{CM}{MA} = -\frac{\beta}{\alpha+\beta}.$$

Multiplying we find

$$\lambda\mu\nu = 1.$$

Ceva's theorem is now proved completely.

Converse. *If we take on each side of a triangle one point (which does not coincide with a vertex) such that the product of the ratios in which these points divide the sides is equal to unity, then the straight lines which join the vertices of the triangle to the points taken on the sides opposite to the vertices pass through one and the same point.*

Shortly: *if* $\lambda\mu\nu = 1$, *then* AL, BM *and* CN *pass through one and the same point.*

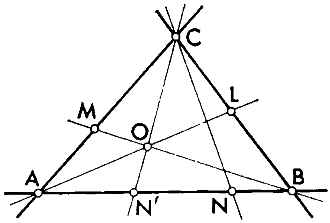


FIG. 30

Proof. Let us assume that the lines AL, BM and CN do not pass through one and the same point (Fig. 30). Denoting by O the point of intersection of AL and BM, draw the straight line CO and denote the point of intersection of CO and AB by N'.

Let us denote the simple ratio (ABN') by ν'. We have

$$\lambda\mu\nu = 1 \text{ (given)},$$
$$\lambda\mu\nu' = 1 \text{ (by Ceva's theorem)}.$$

Hence $\nu = \nu'$. This contradicts the assumption that N and N' are different points. The theorem is proved.

In what follows we shall give the name "Ceva's theorem" to any of the two theorems. They can be combined by using the following formulation:

The necessary and sufficient condition for the passing of straight lines AL, BM *and* CN *through one and the same point is that* $\lambda\mu\nu = 1$.

If we had not introduced ideal points the two theorems would not prove so simple. The first would have to be formulated as follows. If the straight lines AL, BM and CN pass through one and the same point then three cases are possible:

either the product of the three ratios λ, μ, ν is equal to unity,

or one of the lines is parallel to the opposite side of the triangle[1] and the two others cut the opposite sides in the ratios whose product is minus unity,

---

[1] If $AL \parallel BC$, then AL is the notation of a straight line, and point L in this case does not exist.

or two of the lines are parallel to their opposite sides and the third line divides the opposite side (the segment) in halves.

The case shown in Fig. 29 is an impossible one since it is given that the lines pass through one and the same point.

The second theorem would be formulated as follows. If $\lambda\mu\nu = 1$ then the straight lines AL, BM and CN either pass through one and the same point or are parallel. Cases like $AL \parallel BC$ are excluded, since the existence of points L, M, N is given.

The introduction of ideal points makes it possible to combine these (seemingly!) various cases in one formulation.

The tendency to combine cases, which have the same form (though differing in their content), is a characteristic feature of mathematics. A mathematician must be able to notice this unity.

The well-known from the school curriculum theorems on the intersection of medians, bisectors and heights are particular cases of the second of the above theorems. Let us demonstrate it.

If AL, BM and CN are medians, then $\lambda = \mu = \nu = 1$. Hence, $\lambda\mu\nu = 1$.

If AL, BM and CN are bisectors, then $\lambda = \frac{c}{b}$, $\mu = \frac{a}{c}$, $\nu = \frac{b}{a}$. Hence, $\lambda\mu\nu = 1$.

If AL, BM and CN are heights, then $\lambda = \frac{\tan C}{\tan B}$, $\mu = \frac{\tan A}{\tan C}$, $\nu = \frac{\tan B}{\tan A}$. Hence, $\lambda\mu\nu = 1$.

This theorem comprises a special case: the right-angled triangle. Ceva's theorem cannot be applied in this case for it requires that points L, M and N should not coincide with the vertices of the triangle. Of course, the theorem saying that the heights of a triangle pass through one and the same point holds in this case too, but in referring to Ceva's theorem the case of a right-angled triangle must be considered separately.

A historical comment. The author of the theorem, Giovanni Ceva, derived it from mechanical considerations. Let three masses m_1, m_2, m_3 be placed at the vertices A, B, C, respectively. Let L, M, N be the centres of gravity of the pairs (m_2, m_3), (m_3, m_1), (m_1, m_2), respectively. The centre of gravity of two material points lies on the line segment which joins these points and divides it in a ratio, which is inversely proportional to the ratio of the masses, i.e.

$$\left.\begin{aligned}
&L \text{ divides the segment } BC \text{ in the ratio } \lambda = \\
&\qquad = \frac{BL}{LC} = \frac{m_3}{m_2}, \\
&M \text{ divides the segment } CA \text{ in the ratio } \mu = \\
&\qquad = \frac{CM}{MA} = \frac{m_1}{m_3}, \\
&N \text{ divides the segment } AB \text{ in the ratio } \nu = \\
&\qquad = \frac{AN}{NB} = \frac{m_2}{m_1}.
\end{aligned}\right\} \quad (11.3)$$

It is clear that $\lambda\mu\nu = 1$.

Statics tells us that the centre of gravity of three points lies on the segment, which joins one of these points to the centre of gravity of the remaining pairs[1]. It follows that the centre of gravity of three masses m_1, m_2, m_3 is a point, which lies on each of the segments AL, BM, CN, i.e. that all three segments have a point in common.

It can be easily proved that if $\lambda\mu\nu = 1$ it is possible to choose the masses m_1, m_2, m_3 such that the conditions (11.3) be satisfied.

Using the notion of the centre of gravity it is possible to obtain much interesting in geometry.[2] This method can be extended to comprise also negative values of λ, μ, ν.

12. Menelaos' theorem[3]. With the notation used above the two following theorems hold true:

1. *If points L, M and N lie in one straight line, then* $\lambda\mu\nu = -1$.

2. *If* $\lambda\mu\nu = -1$, *then points L, M and N lie in one straight line.*

The reader should remember that the vertices of the triangle ABC are always assumed to be ordinary points, and L, M, N can be ordinary and ideal.

---

[1] This is true for any set of material points, broken up into two subsets.

[2] See M.B. Balk, *Geometric Applications of the Notion of the Centre of Gravity*, Moscow, 1959 (in Russian).

[3] Menelaos of Alexandria, Greek geometer, flourished 98 A.D.

A straight line cutting the sides of a triangle is called (with respect to this triangle) *a transversal*. If the transversal does not pass through any vertex of the triangle it cuts either two sides internally and the third one externally (Fig. 31*a*) or all three sides externally (Fig. 31*b*). The points at which the transversal intersects the sides BC, CA and AB will be designated L, M and N, respectively.

Let us give the proof of the first theorem. Draw through the vertices of the triangle straight lines parallel to the transversal (in Fig. 31 only one of them is drawn, BM'). If the transversal is not parallel to a side of the triangle,

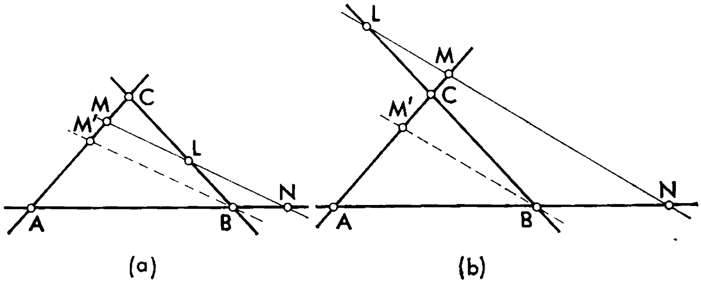


FIG. 31

then the three lines are *different* lines. All ratios, which are of interest to us will be replaced by ratios of line segments on one of the sides of the triangle, for example, on CA.

Denoting by M' the point at which the parallel line which passes through B intersects the side CA we have (note—in all the proportions take account of their signs)

$$\left.\begin{aligned} \frac{BL}{LC} &= \frac{M'M}{MC}, \\ \frac{AN}{NB} &= \frac{AM}{MM'}. \end{aligned}\right\} \qquad \text{(a)}$$

Using (a) we compute the product $\lambda\mu\nu$

$$\lambda\mu\nu = \frac{BL}{LC}\cdot\frac{CM}{MA}\cdot\frac{AN}{NB} = \frac{M'M\cdot CM\cdot AM}{MC\cdot MA\cdot MM'} =$$

$$= \frac{M'M}{MM'}\cdot\frac{CM}{MC}\cdot\frac{AM}{MA} = (-1)\cdot(-1)\cdot(-1) = -1.$$

If the transversal is parallel to one of the sides (Fig. 32) we have a simpler case

$$\frac{AN}{NB}=\frac{AM}{MC}=\frac{MA}{CM}=\frac{1}{\frac{CM}{MA}},$$

i.e. $\nu=\frac{1}{\mu}$ or $\mu\nu = 1$. In this case point L is an ideal one, i.e. $\lambda = -1$. Hence $\lambda\mu\nu = -1$.

Lastly we have to consider one more case, when the transversal is an ideal straight line. In this case all three points L, M, N are ideal, $\lambda = \mu = \nu = -1$ and $\lambda\mu\nu = -1$. The first theorem is completely proved.

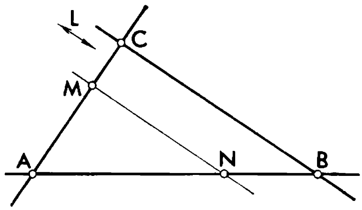


FIG. 32

Let us prove the second one. Given that $\lambda\mu\nu = -1$. Assume that points L, M and N do not lie in one straight line. Let us take the straight line LM to be the transversal and denote by N' the point of intersection of LM and AB and by ν' the simple ratio (ABN'). We have

$\lambda\mu\nu = -1$ (given),
$\lambda\mu\nu' = -1$ (according to the preceding theorem).

Hence $\nu = \nu'$. This contradicts the assumption that N and N' are different points. The theorem is proved.

CHAPTER II

Cross Ratio

13. Notion of a cross ratio. Take a segment AB on a straight line and two points of division C and D (the points must be ordered: for instance, C is taken to be the first and D the second one). Then we obtain two simple ratios:

point C divides segment AB in the ratio $\lambda = (ABC)$,
point D divides segment AB in the ratio $\mu = (ABD)$.

The ratio of these two ratios is called a *cross* (*or double*) *ratio* and denoted by the symbol $(ABCD)$:

$$w = (ABCD) = \frac{\lambda}{\mu} = \frac{(ABC)}{(ABD)}. \qquad (13.1)$$

Interpreting the meaning of the simple ratio we can obtain a direct definition of a cross ratio:

$$w = (ABCD) = \frac{AC}{CB} : \frac{AD}{DB} = \frac{AC \cdot DB}{CB \cdot AD}, \qquad (13.2)$$

or, denoting the points by digits,

$$(1234) = \frac{13}{32} : \frac{14}{42} = \frac{13 \cdot 42}{32 \cdot 14} \qquad (13.3)$$

(of course "13" denotes a segment directed from point 1 to point 3 with a chosen direction of the straight line).

It must be stressed that in the symbol $(ABCD)$ each point plays its particular role:

$\begin{cases} A \text{—is the initial point of the segment,} \\ B \text{—is the end point of the segment,} \end{cases}$

$\begin{cases} C \text{—is the first point of division,} \\ D \text{—is the second point of division.} \end{cases}$

It is expedient to combine the points into pairs as shown above by braces.

It will be made clear in Sec. 15 that the pairs are equivalent, i.e. we may take C and D to be the initial and end points of the segment, and A and B—the first and second points of division.

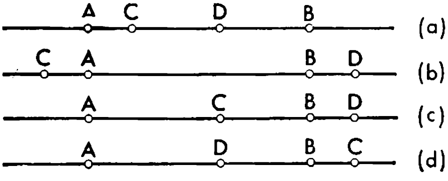


FIG. 33

In nearly all cases we shall assume all these four points to be different. The points of division can be arranged with respect to the segment AB in four ways (Fig. 33):

(a) the two points are inside the segment,
(b) the two points are outside the segment,
(c) the first point is inside, the second outside,
(d) the first point is outside, the second inside.

We now determine the sign of the cross ratio in the above cases:

(a) $\lambda > 0$, $\mu > 0$; $w > 0$,
(b) $\lambda < 0$, $\mu < 0$; $w > 0$,
(c) $\lambda > 0$, $\mu < 0$; $w < 0$,
(d) $\lambda < 0$, $\mu > 0$; $w < 0$.

Thus, the cross ratio is positive if the two points of division are "equally" arranged with respect to the segment and negative if these points are arranged in a different way; as was explained in Sec. 10 this can be formulated as follows:

If the pairs of points do not separate one another the cross ratio is positive, and if they do—it is negative.

It is interesting to find what the cross ratio is if any two points coincide. Let us limit ourselves to cases when the fourth point coincides with one of the three others. Substituting points A, B, C one after the other for point D in formula (13.2) we obtain:

$$\left.\begin{aligned}(ABCA) &= \infty,\\ (ABCB) &= 0,\\ (ABCC) &= 1.\end{aligned}\right\} \qquad (13.4)$$

14. Invariant property of a cross ratio with respect to central projection. The central projection differs from the parallel one (see Fig. 10, p. 19) in that the projectors (projecting straight lines) are not parallel and pass through one and the same point (ordinary) called *the centre of projection*. With parallel projection from one straight line onto the other the lengths of segments are changed, but their ratios remain constant (unchanged). With central projection not only the lengths are changed but their ratios

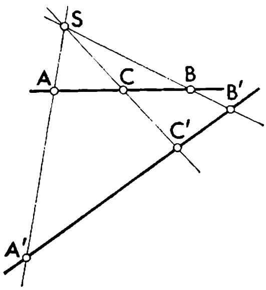


FIG. 34

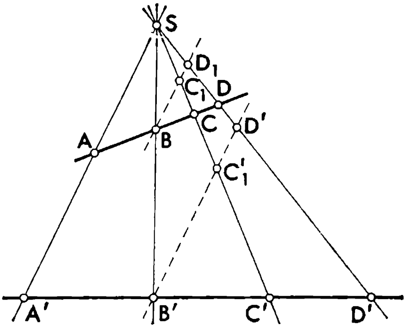


FIG. 35

too. For example, in Fig. 34 point C is the midpoint of the segment AB, but C' is not the midpoint of the segment $A'B'$. Even the sign of the ratio can change. For example, $(ABC) > 0$ (see Fig. 21), but when projected from S onto the straight line a' we obtain $(A'B'C') < 0$. Nevertheless, *the ratio of the two ratios does not change.*

If four points on a straight line are projected onto another straight line their cross ratio remains unchanged[1].

Do not think that there is a misprint: the word "central" has been omitted. The theorem holds true for both kinds of projection—parallel or central. For parallel projection the statement is trivial: since the two separate simple ratios are invariant, their ratio must be invariant too.

[1] In German "werfen" means to throw, "der Wurf"—a throw. The German geometer K.G. Standt (1798-1867) used this term for the cross ratio on the ground that the ratio does not change in "throwing" a tetrad of points from one straight line onto another. This is why the cross ratio is often denoted by w.

Proof. Draw straight lines parallel to a projecting line, SA, say, through two corresponding points, for example B and B'; mark the points at which these two straight lines intersect the other two projectors (Fig. 35). We have:

$$\frac{\overline{AC}}{\overline{CB}} = \frac{\overline{SA}}{\overline{C_1B}},$$

$$\frac{\overline{AD}}{\overline{DB}} = \frac{\overline{SA}}{\overline{D_1B}},$$

$$\frac{\overline{AC}}{\overline{CB}} : \frac{\overline{AD}}{\overline{DB}} = \frac{\overline{D_1B}}{\overline{C_1B}}. \qquad \text{(a)}$$

In the same way we obtain:

$$\frac{\overline{A'C'}}{\overline{C'B'}} : \frac{\overline{A'D'}}{\overline{D'B'}} = \frac{\overline{D_1'B'}}{\overline{C_1'B'}}. \qquad \text{(b)}$$

The right-hand sides of (a) and (b) are equal (for the triangles SBD_1 and $SB'D_1'$ are similar). Consequently, the left-hand sides are equal too

$$\frac{\overline{AC}}{\overline{CB}} : \frac{\overline{AD}}{\overline{DB}} = \frac{\overline{A'C'}}{\overline{C'B'}} : \frac{\overline{A'D'}}{\overline{D'B'}}.$$

Noticing that the two cross ratios both have similar signs we can write:

$$\frac{AC}{CB} : \frac{AD}{DB} = \frac{A'C'}{C'B'} : \frac{A'D'}{D'B'},$$

or

$$(ABCD) = (A'B'C'D'). \qquad (14.1)$$

Q.E.D.

This property can be considered from another point of view:

Four straight lines a, b, c, d of a central pencil cut out on any straight line (which does not pass through the centre of the pencil) the same cross ratio.

Since a cross ratio is independent of the cutting line it belongs to the same tetrad of lines a, b, c, d. This makes it possible to introduce the notion of a cross ratio of four straight lines of a central pencil.

The cross ratio of an ordered tetrad of straight lines of a central pencil is the cross ratio of the four points, which are cut out by these lines on any cutting straight line (of course, if the line does not pass through the centre of the pencil).

The cross ratio of four straight lines is denoted by the symbol $(abcd)$.

The cross ratio of four straight lines can be expressed in terms of the angles formed by these lines, i.e. without recourse to a cutting line. It is left to the reader to prove the following formula:

$$(abcd) = \frac{\sin(a, c)}{\sin(c, b)} : \frac{\sin(a, d)}{\sin(d, b)}. \tag{14.2}$$

Formula (14.2) will not be used in this booklet.

15. Permutation of elements in a cross ratio. In studying (Sec. 7) the permutation of elements in a simple ratio, we found straightway the values of a simple ratio for all the six ways in which three elements can be ordered. With four elements there are twenty-four ways of ordering and to find the values for each of them would be a tedious task. Therefore we shall prove four general rules. Recall that the tetrad $ABCD$ consists of two pairs: AB and CD.

Rule 1. *A cross ratio is not changed when the two pairs are interchanged.*

Indeed, by formula (13.3)

$$(CDAB) = \frac{CA \cdot BD}{AD \cdot CB}.$$

Formula (13.2) shows that this coincides with $(ABCD)$:

$$(CDAB) = (ABCD).$$

Rule 2. *A cross ratio does not change if the elements in each of the two pairs are interchanged simultaneously.*

By formula (13.3)

$$(BADC) = \frac{BD \cdot CA}{DA \cdot BC} = (ABCD).$$

Clearly the cross ratio will not change if the two permutations are performed one after the other. The sequence in which they are performed proves to be of no consequence: in both cases starting with $(ABCD)$ we come to $(DCBA)$ (when the elements are written down in reversed order):

$$(DCBA) = (ABCD).$$

Thus

$$w = (ABCD) = (CDAB) = (BADC) = (DCBA). \tag{15.1}$$

Rules 1 and 2 force us to change the names of the roles played by elements which we introduced in Sec. 12. We cannot treat as different "the initial point and the end point of a segment" and "the points of division" since these pairs are equivalent. From now on we shall say that the cross ratio $(ABCD)$ is formed by *two pairs* AB and CD, without giving different names to the pairs. According to Rule 2 we must regard these pairs as *unordered*.

The sequence of the elements in a pair is of no consequence, but *there is a correspondence between the elements of the pairs*: in a cross ratio $(ABCD)$ element C of the second pair corresponds to element A of the first pair, and element D corresponds to element B. This means that if the sequence of the first pair is taken to be AB, then that of the second must be CD, and if the first pair is BA, then the second one is DC.

Rule 3. *If elements are interchanged only in one pair, then the value of the cross ratio is changed to its inverse.*

Indeed,

$$(BACD)=\frac{BC\cdot DA}{CA\cdot BD}=\frac{1}{w}.$$

Using formula (15.1) we can write

$$\frac{1}{w}=(BACD)=(CDBA)=(ABDC)=(DCAB).$$

Rule 4. *If non-corresponding elements of different pairs are interchanged, then the value of a cross ratio is its difference from unity.*

Indeed, interchanging in the cross ratio $(ABCD)$ the elements B and C, we have

$$(ACBD)=\frac{AB\cdot DC}{BC\cdot AD}.$$

Comparing this expression with (13.2) we find in the numerator segments, which are absent in (13.2). In order to overcome this difficulty let us divide in two parts segment AB by point C, and segment DC—by point B:

$$(ACBD)=\frac{-(AC+CB)(DB+BC)}{CB\cdot AD}=$$

$$=-\frac{AC\cdot DB+AC\cdot BC+CB\cdot DB+CB\cdot BC}{CB\cdot AD}.$$

Let us interchange the letters in the notation of the two segments in the second factor of the numerator and then factor out CB of the three last terms:

$$(ACBD) = -\frac{AC\cdot DB + CB(DB + BC + CA)}{CB\cdot AD} =$$

$$= -\frac{AC\cdot DB + CB\cdot DA}{CB\cdot AD} = -\frac{AC\cdot DB}{CB\cdot AD} + 1 = 1 - w.$$

Four elements can be permuted in 24 ways. Our four rules make it possible to determine the values of the corresponding cross ratios without any further computations:

$$1 - w = (ACBD) = (BDAC) = (CADB) = (DBCA).$$

Applying Rule 3 we have

$$\frac{1}{1-w} = (CABD) = (BDCA) = (ACDB) = (DBAC).$$

Applying Rule 4 to $\frac{1}{w}$ we obtain:

$$\frac{w-1}{w} = (BCAD) = (ADBC) = (CBDA) = (DACB).$$

Applying Rule 3 to the last cross ratios we have:

$$\frac{w}{w-1} = (CBAD) = (ADCB) = (BCDA) = (DABC).$$

We have exhausted all the 24 permutations of four elements. We give below a summary of the results obtained substituting digits for letters in the notation of the elements.

$$\left.\begin{aligned} w &= (1234) = (3412) = (2143) = (4321), \\ \frac{1}{w} &= (2134) = (3421) = (1243) = (4312), \\ 1 - w &= (1324) = (2413) = (3142) = (4231), \\ \frac{1}{1-w} &= (3124) = (2431) = (1342) = (4213), \\ \frac{w-1}{w} &= (2314) = (1423) = (3241) = (4132), \\ \frac{w}{w-1} &= (3214) = (1432) = (2341) = (4123). \end{aligned}\right\} \quad (15.2)$$

Notice that there is one essential difference between this result and that of Sec. 7 (see table (7.2)). Six permutations can be made with three points in a straight line and, *generally speaking, a different simple ratio corresponds to each of them.*

24 permutations are possible with four points in a straight line and, *generally speaking, only six different cross ratios correspond to them*[1]. The 24 permutations break up into six tetrads with equal cross ratios in each of them.

16. Harmonic tetrads. Are there any special arrangements of four points in a straight line with less than six values of cross ratios, which correspond to these arrangements? In order to answer this question we must find the values of w for which some of the left-hand sides of formulas (15.2) coincide.

Remark 1. It is sufficient to equate one of the left-hand sides, w say, to each of the rest. The reason for this is the same as the one given in Sec. 7.

Remark 2. We seek a tetrad of points, without fixing their roles (i.e. all the points are equivalent and cannot be designated either by letters, or digits or in any other way). Such a tetrad is characterized not only by one but by six values of the cross ratio.

Rermark 3. We assume that all the four points are different, i.e. we seek a real tetrad and not a triad. If two of the points coincide the problem becomes a trivial one, since it is clear beforehand that interchanging points which coincide changes nothing.

We are going now to carry out our plan.

(1) $w = \frac{1}{w}$, $w^2 = 1$, $w = \pm 1$.

The value $w = 1$ must be rejected since it corresponds to a degenerated tetrad (see formulas (13.4)). We retain the value $w_1 = -1$.

(2) $w = 1 - w$, $w_2 = \frac{1}{2}$.

(3) $w = \frac{1}{1-w}$, $w^2 - w + 1 = 0$. Imaginary roots.

(4) $w = \frac{w-1}{w}$, $w^2 - w + 1 = 0$. The same as above.

(5) $w = \frac{w}{w-1}$. The value $w = 0$ corresponds to a degenerated tetrad. We retain the value $w_3 = 2$.

For each of the found values of w we list the six corresponding values:

---

[1] With a certain special arrangement of the points—even less (see Sec. 16).

w	$\frac{1}{w}$	$1-w$	$\frac{1}{1-w}$	$\frac{w-1}{w}$	$\frac{w}{w-1}$
-1	-1	2	$\frac{1}{2}$	2	$\frac{1}{2}$
$\frac{1}{2}$	2	$\frac{1}{2}$	2	-1	-1
2	$\frac{1}{2}$	-1	-1	$\frac{1}{2}$	2

In all the rows we have the same numbers. This means that the three found values of w should be considered as representing the same solution: they correspond to the same tetrad of points, ordered in different ways. Such a tetrad is called *harmonic* (below we shall give a more direct formulation of the definition of a harmonic tetrad).

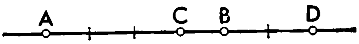


FIG. 36

What does a harmonic tetrad look like? Any tetrad of points $ABCD$ in a straight line can be broken into two pairs in three ways: (1) AB and CD, (2) AC and BD, (3) AD and BC. With any arrangement of the points one of these ways gives separated pairs and the other two non-separated ones[1].

Let us break the harmonic tetrad into two separated pairs. Such a break corresponds to the value $w = -1$. Recall that $w = \frac{\lambda}{\mu}$. It follows that $\lambda = -\mu$. It is now clear how a harmonic tetrad can be represented. Take a segment AB and a point C inside it, which divides it in a certain ratio λ, and outside it a point D, which divides the segment in **the same** (as to its absolute value) ratio.

Example. In Fig. 36 (a scale is marked in it, for convenience of use) point C divides the segment AB in the ratio $\lambda = 3$ and point D divides the same segment in the ratio $\mu = -3$. The tetrad $ABCD$ in Fig. 36 is a harmonic set.

[1] The algebraic equivalent of this statement is as follows: of the six numbers w, $\frac{1}{w}$, $1 - w$, $\frac{1}{1-w}$, $\frac{w-1}{w}$, $\frac{w}{w-1}$ four are positive and two negative.

We shall state now several properties of a harmonic tetrad. Their substantiation is left to the reader.

1. In a harmonic tetrad as in any other the pairs are equivalent. This means that we can take C and D to be the extreme points of the segment and A and B to be the points, which divide the segment in the same (as to its absolute value) ratio internally and externally.

For example, in Fig. 36:

point A divides the segment CD in the ratio $\lambda_1 = -\frac{1}{2}$,

point B divides the segment CD in the ratio $\mu_1 = \frac{1}{2}$.

However a harmonic tetrad differs from any other in that there is no correspondence in it between the points of the two pairs, i.e. point C cannot be taken to correspond to point A. Indeed, with permutation of the elements in one pair the cross ratio $w = -1$ will be changed to its inverse, i.e. will not change. For example, in Fig. 36:

FIG. 37

point C divides the segment BA in the ratio $\lambda_3 = \frac{1}{3}$,

point D divides the segment BA in the ratio $\mu_3 = -\frac{1}{3}$.

A harmonic tetrad consists of two separate pairs. Each pair is unordered. The pairs are equivalent. The points of one pair are conjugate with respect to the other pair.

2. To make this point palpable: if C inside the segment is close to A, then D outside the segment is close to A too. If C moves to the right, then D begins to move to the left (Fig. 37). When C reaches the midpoint, then D becomes the ideal point ($\lambda = 1$, $\mu = -1$). This fact should be especially noted.

The midpoint of the segment is harmonically conjugated with the ideal point.

When C is to the right of the midpoint, D appears on the right and will move towards C in the opposite direction. Points C and D will approach point B from different sides.

Let us consider now what harmonic tetrads of straight lines look like. In order to obtain a harmonic tetrad of straight lines a harmonic tetrad of points must be projected from some point S. Take an arbitrary triangle ABS (Fig. 38) and mark on its base AB a harmonic tetrad of points: (1) the vertices A and B, (2) the midpoint of the side C and the ideal point D. Projecting this tetrad from the vertex S we find:

For each vertex of a triangle there is a harmonic tetrad of straight lines: (1) *two sides of the triangle*, (2) *a median and a line parallel to the base.*

This proposition enables us to draw easily a harmonic tetrad of straight lines. Cutting it by various straight lines,

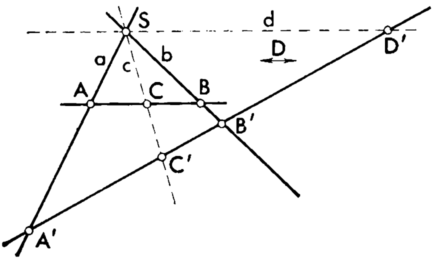


FIG. 38

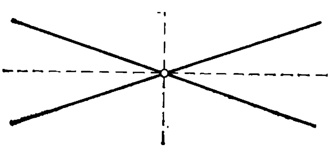


FIG. 39

various harmonic tetrads of points can be obtained: see, for example, the tetrad $A'B'C'D'$ in Fig. 38.

Another simple method of obtaining harmonic tetrads of straight lines can be given:

Two intersecting straight lines and their two bisectors form a harmonic tetrad (Fig. 39).

17. Finding the fourth point given a cross ratio. Section 4 showed how to solve the problem: given two points A, B and the simple ratio $\lambda = (ABC)$, find the third point C. It is natural to pose an analogous problem for the cross ratio: *having three points of the tetrad $ABCD$ in a straight line and knowing the cross ratio $w = (ABCD)$ find the fourth point.*

We shall first solve an auxiliary problem.

Given three straight lines of a central pencil a, b, c and three points in a straight line A_0, B_0, C_0 (Fig. 40). *Draw a line on which the straight lines a, b, c cut out a triad of points A, B, C which is congruent with A_0, B_0, C_0.*

It is expedient to start with plotting the points A_0, B_0, C_0 and then to construct the angles between the lines a,

b, c. Point S_0 must lie on the arc $A_0S_0B_0$ which contains the angle (a, b) (Fig. 41). There are two such arcs. For definiteness we shall take one of them. Point S_0 must lie at the same time on the arc $B_0S_0C_0$ which contains the angle (b, c) (we take this arc on the side of straight line A_0B_0 where the arc $A_0S_0B_0$ lies). Point S_0 will be the second (besides B_0) point of intersection of two circles. The case of two circles touching one another cannot occur as the condition for this is that $(a, b) + (b, c) = 180°$. We now turn back to Fig. 40

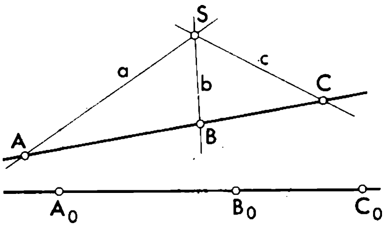


FIG. 40

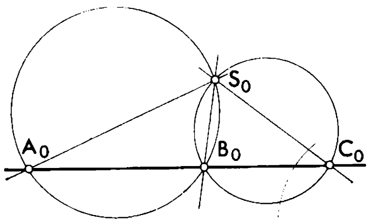


FIG. 41

and mark off on the straight lines a, b, c from point S the segments $SA = S_0A_0$, $SB = S_0B_0$, $SC = S_0C_0$, arranged as in Fig. 41. The straight line ABC is the line we sought. There are two solutions to the problem. The second straight line ABC is symmetrical to the first one, with respect to point S.

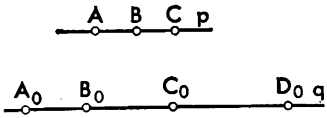


FIG. 42

If one of the points, for example, C is ideal, then the problem has a simpler solution: we must cut the triad a, b, c by a straight line, which is parallel to c and then replace the drawing by a similar one.

We now pass on to the main problem.

Three points A, B, C are given in a straight line p (Fig. 42), and four points A_0, B_0, C_0, D_0—in line q. This tetrad gives in graphic form the cross ratio $w = (A_0B_0C_0D_0)$. It is required to find point D such that it forms with the triad ABC the same cross ratio.

Solution. From an arbitrary point S project the given points A, B, C. Cut the triad of lines $a = SA$, $b = SB$, $c = SC$ by a straight line so as to obtain a triad of points

$A_0'B_0'C_0'$ (Fig. 43) which is congruent with $A_0B_0C_0$. We now plot point D_0' (the tetrad $A_0'B_0'C_0'D_0'$ is congruent with $A_0B_0C_0D_0$). Join D_0' to S. The point at which $D_0'S$ cuts line p is the point we seek. It is obvious that there is only one solution.

Let us now consider the particular and moreover most interesting case of a harmonic tetrad. Given a triad of points

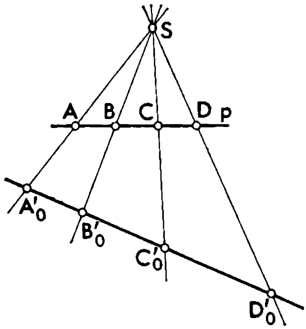


FIG. 43

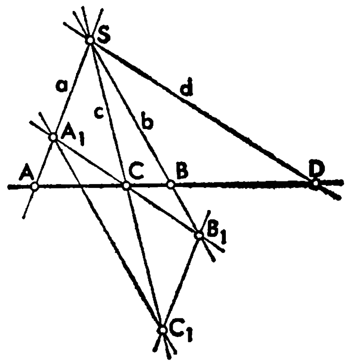


FIG. 44

in a straight line, it is required to supplement it by a fourth point such that we obtain a harmonic tetrad. There is no need to order the triad given; we have only to point out which two points are a pair. Given are three points ABC, and it is known, that points A and B, for example, are a pair, and C is one point of the other pair. It is required to find point D such that $(ABCD) = -1$. The problem can be formulated shortly thus:

Find point D, which is harmonically conjugated with C with respect to A and B.

This problem can be solved by the above method and also by many simpler ones. We give here two such methods, based on the properties of harmonic pencils (Figs. 38 and 39).

1. Project the three given points ABC from an arbitrary point S (Fig. 44). We now have to draw through point C a straight line whose segment inside the angle ASB would divide in halves at point C.

For this purpose mark off $CC_1 = SC$ and draw $C_1A_1 \parallel BS$ and $C_1B_1 \parallel AS$. The figure $SA_1C_1B_1$ is a parallelogram. Its diagonals at the point of intersection cut each other into halves. Consequently, SC is a median of triangle A_1B_1S. It remains to draw through S a straight line parallel to A_1B_1. This line cuts AB at point D, which is the point sought.

2. Draw a circle through points A and B (Fig. 45). Find the midpoint C_1 of the arc AB (no matter which of the two). Draw the straight line C_1C and mark S, the point of its intersection with the circle. The straight line $c \equiv SC$ is the bisector of the angle (a, b). It remains to draw the bisector d of the other angle (adjoining) and d will cut AB at the point sought, D.

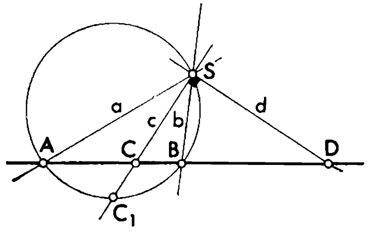


FIG. 45

These methods are rule of thumb and require laborious investigations of all possible particular cases. We shall not take up this investigation, and are going to show a much better way. It is independent of the arrangement of elements and of the existence among the given points of an ideal one. Besides, its application requires the use of only a ruler.

18. Theorem on a complete quadrilateral. A complete quadrilateral is a *plane* figure formed by the following elements: (1) four points of general position (this means that no three of them lie in one line), (2) six straight lines, joining these points in pairs[1]. The four points are called *vertices* of the complete quadrilateral, and the six straight lines its *sides*.

The term, quadrilateral, is used to avoid using the word "quadrangle", which can give rise to familiar associations. The quadrilateral is not a part of a plane—it has no inside.

A complete quadrilateral is shown in Fig. 46. Its vertices are marked by small circles and designated by the letters A, B, C, D; this does not mean that the vertices are ordered. The vertices of a complete quadrilateral are equivalent. it has no "next" or "opposite" vertices.

The sides of a complete quadrilateral intersect, besides the vertices, in three more points.

These points are called *diagonal points*.

---

[1] A simple quadrilateral is an *ordered* tetrad of points and *four* straight lines, which join the points in *consecutive order*, i.e. 1 to 2, 2 to 3, 3 to 4 and 4 to 1.

In Fig. 46 they are marked by small squares and designated by P, Q, R.

Consider the straight line joining two diagonal points, for example, PQ. Two sides of the quadrilateral intersect at each of these points. A quadrilateral has two more sides. They intersect the straight line PQ (or QR, or RP) at two points. All such points are marked in Fig. 46 by small triangles. It is found that two "squares" and two "triangles" form a harmonic tetrad.

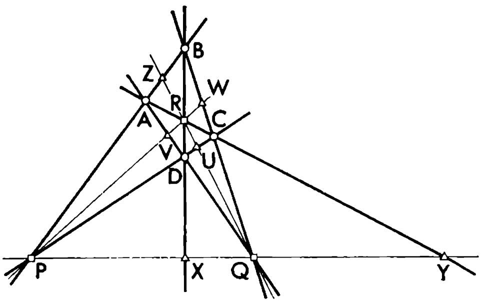


FIG. 46

Theorem on a complete quadrilateral. *The two points at which the straight line joining a pair*[1] *of diagonal points of a complete quadrilateral is cut by the two remaining sides of the quadrilateral form with this pair of diagonal points a harmonic tetrad.*

Proof. Project the tetrad $PQXY$ from point A onto the straight line BD. The points P, Q, X, Y when projected will become B, D, X, R, respectively. A cross ratio does not change with central projection

$$w = (PQXY) = (BDXR). \qquad \text{(a)}$$

Project the tetrad $(BDXR)$ from point C back onto the line PQ. When projected it becomes the tetrad $QPXY$:

$$(BDXR) = (QPXY). \qquad \text{(b)}$$

It follows from (a) and (b) that

$$w = (PQXY) = (QPXY). \qquad (18.1)$$

[1] Any pair.

If $(PQXY) = w$, then $(QPXY) = \frac{1}{w}$ (see (15.2)). Thus, $w = \frac{1}{w}$; hence $w = \pm 1$. However w cannot be equal to unity, since P, Q, X, Y are all different points. Therefore $w = (PQXY) = -1$, i.e. $PQXY$ is a harmonic tetrad, Q.E.D.

This reasoning is applicable to analogous tetrads $QRZU$ and $RPVW$. Moreover, it has been found in the process of reasoning that the tetrad $BDRX$ is a harmonic one too. This applies also to analogous tetrads $ABPZ$, $ACRY$, $ADQV$, $BCQW$ and $CDPU$.

The proof will change in nothing if some of the points prove ideal.

This theorem makes it possible to find the fourth harmonic point by completing with the use of the given triad the construction of a complete quadrilateral. The steps of this construction are shown in Fig. 47.

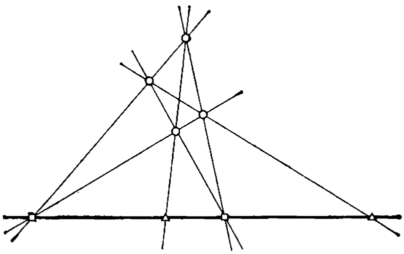


FIG. 47

(1) Draw two straight lines through one of the "squares". (2) Draw a line through the "triangle". Its intersections with the first two lines determine two of the vertices of the complete quadrilateral ("circles"). (3) Join the "circles" to the other "square". This determines the remaining two "circles". (4) Draw the last side (join the last two "circles"). This line cuts the given straight line at the second "triangle".

Exercise. Using the theorem on the complete quadrilateral, prove that the line which joins the point of intersection of the lateral sides of a trapezium to that of the diagonals divides the parallel sides of the trapezium in halves. Compare this with the elementary proof.

19. Group property of a cross ratio. The expressions on the left-hand sides of the formulas (15.2) possess a group property, for the same reasons as those in Sec. 8: if any of the expressions is substituted in any other for w we obtain again one of them.

Let us use the notation:

$$\left.\begin{aligned} a_1 = w,\ a_2 = \frac{1}{w},\ a_3 = 1 - w, \\ a_4 = \frac{1}{1-w},\ a_5 = \frac{w-1}{w},\ a_6 = \frac{w}{w-1}. \end{aligned}\right\} \quad (19.1)$$

We define "multiplication" as in Sec. 8: "To multiply" a_i by a_j means to substitute a_j in a_i for w. For example,

$$a_3 \odot a_4 = 1 - \frac{1}{1-w} = \frac{w}{w-1} = a_6.$$

Performing these operations with various pairs we obtain the "multiplication" table ("Cayley square"). The reader should draw up the table by himself. He will be probably surprised to find that his table coincides with the table (8.4).

The groups (8.1) and (19.1) from a geometric point of view are different. One consists of simple ratios, the other of cross ones. If we pay no attention to the concrete meaning of the elements and consider only their interrelation in the "multiplication" there is no difference between them. This reminds us of the fact, that from the viewpoint of abstract arithmetic there is no difference between the equalities "2 apples and 3 apples = 5 apples" and "2 pencils and 3 pencils = 5 pencils", though an apple and a pencil are not the same thing. Arithmetic tells us that $2 + 3 = 5$ abstracting from the nature of the objects being added. In just the same way a specialist in group theory considers the group of simple ratios and that of cross ratios to be one and the same group. This group has many other concrete contents (for instance, the group of substitutions of three elements — see the books of P.S. Aleksandrov and I. Grossman and W. Magnus, referred to on page 25).

PROBLEMS

We have come to the end of the booklet. If its subject attracted the reader's interest, let him solve a few problems. Some of them are considerably more difficult than the matter expounded in the booklet.

1. Given segment AB. Find in AB a pair of points CD, which divide AB harmonically. Given besides (a) $\overline{CD} = 3/4\ \overline{AB}$, (b) $\overline{CD} = K \cdot \overline{AB}\ (K > 0)$ ($\overline{AB}$ is the length of segment AB).

Problems on "division". What is meant by "divide b by a"? It means that we have to solve the equation $ax = b$. Since for the group (8.4) the "multiplication" is not commutative, it is natural to define two different "divisions", viz: "left division" of b by a means to solve the equation $a \odot x = b$, and "right division" of b by a means to solve the equation $y \odot a = b$ (a, b, x and y denote the elements of a group, a and b are given, x and y are the unknowns).

2. Perform the "left division" of a_5 by a_3.

3. Perform the "right division" of a_5 by a_3.

4. Prove that the "division" (right or left) of a_i by a_j is always (with any elements a_i and a_j) possible and the solution is single-valued.

An element of group a_i is termed *cyclic* if there is a natural number m such that $a_i^m = a_1$ (the m-th power of a is defined as the "product" of m factors $a_i \odot a_i \odot \ldots \odot a_i$, a_1 is a unit element). If $a_i^n \neq a_1$ with $n = 1, 2, \ldots, m-1$ (i.e. if m is the *least* exponent with which $a_i^m = a_1$), then m is *the order* of the cyclic element a_i.

5. Prove that all the elements of group (8.4) are cyclic and find their orders.

6. Compute (a) $a_2^3 \odot a_4^2 \odot a_5$, (b) $a_3^{25} \odot a_6^3$.

7. *The connection between Ceva's and Menelaos' theorems.* Two points are taken on each side of the triangle ABC (the sides are straight lines of indefinite extent), which separate harmonically the vertices of the triangle: on side BC — points L and L', on CA — M and M', on AB — N and N'. To prove that: (a) if straight lines AL, BM and CN pass through one and the same point U, then points L', M' and N' lie in one straight line u (straight line u is called *the harmonic polar* of point U with respect to the triangle

ABC). (b) Conversely: if points L', M' and N' lie in one straight line u, then lines AL, BM and CN pass through one and the same point U (point U is called *the harmonic pole* of the straight line u with respect to the triangle ABC).

In Problems 8 through 10 the triangle ABC is considered with points L, M and N on its sides, the straight lines AL, BM and CN not necessarily passing through one and the same point. In this case they form a triangle $A'B'C'$ (Fig. 48). The vertices of the triangle $A'B'C'$ are projected onto the opposite sides as pairs of points LL', MM' and NN'.

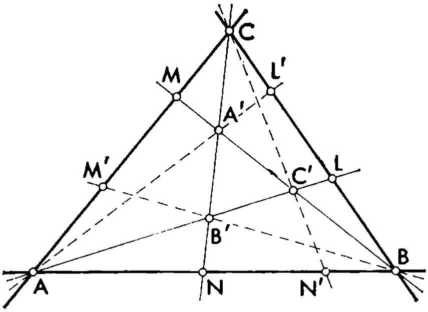


FIG. 48

The simple ratios in which points L, M and N divide the sides of the triangle ABC are denoted as before by λ, μ and ν, and the analogous ratios for points L', M' and N' — by λ', μ' and ν'.

8. Prove (Izvolsky's theorem, 1929) that $(BCLL') = (CAMM') = (ABNN') = \lambda\mu\nu$.

Derive from this Ceva's theorem.

9. Prove (Routh's theorem, 1896) that the ratio of the areas of the triangles $A'B'C'$ and ABC is expressed as follows:

$$\frac{S'}{S} = \frac{(\lambda\mu\nu - 1)^2}{(1+\lambda+\lambda\mu)(1+\mu+\mu\nu)(1+\nu+\nu\lambda)}.$$

Derive from this Ceva's theorem.

10. Prove that the ratio of the areas of the triangles LMN and ABC is expressed as follows:

$$\frac{S_0}{S} = \frac{\lambda\mu\nu + 1}{(1+\lambda)(1+\mu)(1+\nu)}.$$

Derive from this Menelaos' theorem.

11. Prove the following theorem, that may be called Ceva's theorem for three-dimensional space.

Given an arbitrary tetrahedron $A_0A_1A_2A_3$.

On each edge A_iA_j (the edge is a straight line of indefinite extent) we take *a supplementary point* A_{ij}, which coincides neither with A_i nor with A_j. If in the triangle $A_iA_jA_k$ the straight lines A_iA_{jk}, A_jA_{ik} and A_kA_{ij} all pass through one and the same point we say *Ceva's phenomenon* is present in this triangle and this point is called *Ceva's point* and denoted by A_{ijk}.

Theorem. If Ceva's phenomenon is present on three faces of the tetrahedron $A_0A_1A_2A_3$, then: (1) it is present on the fourth face, too; (2) the seven straight lines A_0A_{123}, A_1A_{023}, A_2A_{013}, A_3A_{012}, $A_{01}A_{23}$, $A_{02}A_{13}$, and $A_{03}A_{12}$ pass through one and the same point (it is natural to designate this point by A_{0123} and call it *Ceva's point* of the tetrahedron).

ANSWERS AND SOLUTIONS

1. Use the second of formulas (4.1) (p. 18)

The problem has two solutions: (a) $\lambda_C = -\lambda_D = 3$ or $\lambda_C = -\lambda_D = \frac{1}{3}$; (b) $\lambda_C = -\lambda_D = \frac{\sqrt{1+k^2}+1}{k}$ or $\lambda_C = -\lambda_D = \frac{\sqrt{1+k^2}-1}{k}$.

2. a_2. **3.** a_6. **4.** That "left division" is possible follows from the fact that all elements of the group are contained in each row of table (8.4), and the single value of the solution is the consequence of each element occurring only once. That "right division" is possible and the solution in this case is single-valued follows from the analogous properties of the columns. **5.** Element a_1 is of the first order, a_2, a_3 and a_6—of the second, a_4 and a_5—of the third. **6.** In solving this problem one must use the results of the preceding one. For example, knowing that a_3 is a cyclic element of the second order, one can exclude the 2-s in the exponent of a_3^{25}: $a_3^{25} = a_3^1 = a_3$. (a) a_3; (b) a_4. **7.** If AL, BM and CN pass through one and the same point, then $\lambda\mu\nu = 1$. But $\lambda' = -\lambda$, $\mu' = -\mu$, $\nu' = -\nu$. If $\lambda\mu\nu = 1$ then $\lambda'\mu'\nu' = -1$ and consequently points L', M' and N' lie in one straight line. The proof of the converse is analogous. **8.** Project the tetrad of points $BCLL'$ from A onto the straight line CN, then the tetrad obtained from B onto CA. These tetrads have equal cross ratios $(BCLL') = (NCB'A') = (ACM'M)$. Interchange the elements in each pair of the last cross ratio (Rule 2, Sec. 15):

$$(BCLL') = (CAMM').$$

In an analogous way one can prove that the third cross ratio is equal to each of the two first ones:

$$(BCLL') = (CAMM') = (ABNN').$$

Now note that the straight lines AL', BM and CN all pass through point A'. So, by Ceva's theorem,

$$\frac{BL'}{L'C} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = 1$$

or

$$\lambda'\mu\nu = 1, \quad \lambda' = \frac{1}{\mu\nu}.$$

Compute the cross ratio $(BCLL')$:

$$(BCLL') = \frac{BL}{LC} : \frac{BL'}{L'C} = \lambda : \lambda' = \lambda : \frac{1}{\mu\nu} = \lambda\mu\nu.$$

If the straight lines AL, BM and CN pass through one and the same point, then point L coincides with L' and consequently $\lambda\mu\nu = (BCLL') = (BCLL) = 1$ (formulas (13.4)).

9. This theorem can be proved easily by the analytical method. The reader who has not begun to study analytical geometry should better leave the proof for the future. We shall take the rays AB and AC as axes of coordinates (oblique-angled system!) and the line segments AB and AC as (unequal!!) scale units. The points of interest to us will have the following coordinates: A (0, 0), B (1, 0), C (0, 1), $L\left(\frac{1}{1+\lambda}, \frac{\lambda}{1+\lambda}\right)$, $M\left(0, \frac{1}{1+\mu}\right)$, $N\left(\frac{\nu}{1+\nu}, 0\right)$. The equation of AL is $\lambda x - y = 0$, the equation of BM is $x + (1 + \mu) y = 1$, the equation of CN is $(1 + \nu) x + \nu y = \nu$. Solving these equations in pairs we obtain:

$$A'\left(\frac{\mu\nu}{1+\mu+\mu\nu}, \frac{1}{1+\mu+\mu\nu}\right),$$

$$B'\left(\frac{\nu}{1+\nu+\nu\lambda}, \frac{\nu\lambda}{1+\nu+\nu\lambda}\right),$$

$$C'\left(\frac{1}{1+\lambda+\lambda\mu}, \frac{\lambda}{1+\lambda+\lambda\mu}\right).$$

The ratio of the areas of the triangles is expressed by using determinants of the third order:

$$S' : S = \begin{vmatrix} x_{A'} & y_{A'} & 1 \\ x_{B'} & y_{B'} & 1 \\ x_{C'} & y_{C'} & 1 \end{vmatrix} : \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix} =$$

$$= \begin{vmatrix} \frac{\mu\nu}{1+\mu+\mu\nu} & \frac{1}{1+\mu+\mu\nu} & 1 \\ \frac{\nu}{1+\nu+\nu\lambda} & \frac{\nu\lambda}{1+\nu+\nu\lambda} & 1 \\ \frac{1}{1+\lambda+\lambda\mu} & \frac{\lambda}{1+\lambda+\lambda\mu} & 1 \end{vmatrix} : \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} =$$

$$= \frac{1}{(1+\lambda+\lambda\mu)(1+\mu+\mu\nu)(1+\nu+\nu\lambda)} \begin{vmatrix} \mu\nu & 1 & 1+\mu+\mu\nu \\ \nu & \nu\lambda & 1+\nu+\nu\lambda \\ 1 & \lambda & 1+\lambda+\lambda\mu \end{vmatrix}.$$

Subtract the sum of the two first columns of the determinant from the third column

$$S' : S = \frac{1}{(1+\lambda+\lambda\mu)(1+\mu+\mu\nu)(1+\nu+\nu\lambda)} \begin{vmatrix} \mu\nu & 1 & \mu \\ \nu & \nu\lambda & 1 \\ 1 & \lambda & \lambda\mu \end{vmatrix} =$$

$$= \frac{(\lambda\mu\nu - 1)^2}{(1+\lambda+\lambda\mu)(1+\mu+\mu\nu)(1+\nu+\nu\lambda)}.$$

If the straight lines AL, BM and CN pass through one and the same point, then $S' = 0$ and consequently $\lambda\mu\nu = 1$.

If the reader is familiar only with Cartesian coordinates (axes at right angles with equal scales on them), he can ascribe to the original points the coordinates A (0, 0), B (a, 0), C (b, c) and repeat the above reasoning, as stated here. The derivation will be more laborious, but the final result will be the same.

It can be seen how important it is to select for every problem the most suitable tools.

10. Having] solved the preceding problem very little remains to be done. The coordinates of points L, M and N are already found, the ratio of the areas is computed in the same way as in the preceding problem. If points L, M and N are in one straight line, then $S_0 = 0$ and consequently $\lambda\mu\nu = -1$.

11. In place of the notation λ, μ, ν we introduce $\lambda_{ij} = \frac{A_iA_{ij}}{A_{ij}A_j}$. Clearly $\lambda_{ij} \cdot \lambda_{ji} = 1$. (a). Assume that Ceva's phenomenon is present on the faces, which pass through the vertex A_0. Then $\lambda_{01} \cdot \lambda_{12} \cdot \lambda_{20} = 1$, $\lambda_{02} \cdot \lambda_{23} \cdot \lambda_{30} = 1$, $\lambda_{03} \times \lambda_{31} \cdot \lambda_{10} = 1$. Multiplying these equalities and using (a) we obtain $\lambda_{12} \cdot \lambda_{23} \cdot \lambda_{31} = 1$, i.e. Ceva's phenomenon is present on the face $A_1A_2A_3$ too.

In order to prove the second part of the theorem let us construct two triads of planes

$$\left.\begin{aligned} \alpha_1^0 &\equiv A_0A_1A_{23}, \\ \alpha_2^0 &\equiv A_0A_2A_{13}, \\ \alpha_3^0 &\equiv A_0A_3A_{12}. \end{aligned}\right\} \quad \text{(b)} \qquad \left.\begin{aligned} \alpha_0^1 &\equiv A_1A_0A_{23}, \\ \alpha_2^1 &\equiv A_1A_2A_{03}, \\ \alpha_3^1 &\equiv A_1A_3A_{02}. \end{aligned}\right\} \quad \text{(c)}$$

All the planes (b) contain point A_{123} and the planes (c)—point A_{023}. This means that the planes (b) pass through the straight line A_0A_{123} and the planes (c) through the straight line A_1A_{023}. α_1^0 and α_0^1 are one and the same plane. Thus the straight lines A_0A_{123} and A_1A_{023} lie in the same plane and consequently intersect (maybe in an ideal point). It can be proved by an analogous reasoning that any two straight lines which join the vertices of the tetrahedron with Ceva's points of the opposite faces intersect. Thus the straight lines A_0A_{123}, A_1A_{023}, A_2A_{013} and A_3A_{012} form intersecting *pairs*. This is possible only in two cases: (1) all four straight lines have a point in common, (2) all four straight lines lie in one plane. The second case is excluded, since these straight lines contain all the vertices of the tetrahedron, i.e. the realization of the second case would mean that all the vertices of the tetrahedron lie in one plane. Only the first case remains, Q.E.D. We denote the point of intersection of the four straight lines by A_{0123}.

It remains to prove that the straight lines which join the supplementary points of opposite edges also pass through point A_{0123}. These three straight lines lie in the planes (b) and consequently intersect the straight line A_0A_{123}. They lie also in the planes (c) and consequently intersect the straight line A_1A_{023}. The last straight lines have only one point in common. This means that the straight lines $A_{01}A_{23}$, $A_{02}A_{13}$ and $A_{03}A_{12}$ either pass through this point, or lie in the plane which contains the straight lines A_0A_{123} and A_1A_{023}. The second possibility must be rejected since points A_{23}, A_{13} and A_{12} lie on the face $A_1A_2A_3$, and points A_{01}, A_{02}, and A_{03} cannot lie in this plane. This means that the straight lines $A_{01}A_{23}$, $A_{02}A_{13}$ and $A_{03}A_{12}$ pass through point A_{0123}, Q.E.D.

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USSR, 129820, Moscow I-110, GSP
Pervy Rizhsky Pereulok, 2
Mir Publishers

Printed in the Union of Soviet Socialist Republics

The booklet describes various theories to which a deep study of the problem of dividing a line segment in a given ratio leads. Considering this elementary and those related to it, the reader will make a short travel over some branches of mathematics come into contact woth affine and projective geometry and the theory of groups, though in most cases without mentioning those names.

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