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E. Thorndike

AN

ELEMENTARY TREATISE

ON

ARITHMETIC,

TAKEN

PRINCIPALLY FROM THE ARITHMETIC

OF

S. F. LACROIX,

AND

TRANSLATED INTO ENGLISH WITH SUCH ALTERATIONS AND ADDITIONS AS WERE FOUND NECESSARY IN ORDER TO ADAPT IT TO THE USE OF THE AMERICAN STUDENT.

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ADVERTISEMENT.

The first principles, as well as the more difficult parts of Mathematics, have, it is thought, been more fully and clearly explained by the French elementary writers, than by the English; and among these, Lacroix has held a very distinguished place. His treatises have been considered as the most complete, and the best suited to those who are destined for a public education. They have received the sanction of the Government, and have been adopted in the principal schools, of France. The following translation is from the thirteenth Paris edition. The original being written with reference to the new system of weights and measures, in which the different denominations proceed in a decimal ratio, it was found necessary to make considerable alterations and additions, to adapt it to the measures in use in the United States. The several articles relating to the reduction, addition, subtraction, multiplication, and division, of compound numbers, have been written anew; a change has been made in many of the examples and questions, and new ones have been introduced after most of the rules, as an exercise for the learner.

JOHN FARRAR,
Professor of Mathematics and Natural Philosophy in the University at Cambridge.

Cambridge, Aug. 1818.

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Ninety	XC
Hundred	C
Two hundred	CC
Three hundred	CCC
Four hundred	CCCC

* As often as any character is repeated, so many times its value is repeated.

† A less character before a greater diminishes its value.

‡ A less character after a greater increases its value.

Five hundred	D or IↃ*
Six hundred	DC
Seven hundred	DCC
Eight hundred	DCCC
Nine hundred	DCCCC
Thousand	M or CIↃ†
Eleven hundred	MC
Twelve hundred	MCC
Thirteen hundred	MCCC
Fourteen hundred	MCCCC
Fifteen hundred	MD
Two thousand	MM
Five thousand	IↃↃ : or $\overline{V}$‡
Six thousand	$\overline{VI}$
Ten thousand	$\overline{X}$ or CCIↃↃ
Fifty thousand	IↃↃↃ
Sixty thousand	$\overline{LX}$
Hundred thousand	$\overline{C}$ or CCCIↃↃↃ
Million	$\overline{M}$ or CCCCIↃↃↃↃ
Two millions	$\overline{MM}$
&c. &c.	

* For every Ↄ affixed this becomes ten times as many

† For every C and Ↄ put one at each end, it is increased ten times.

‡ A line over any number increases it 1000 fold.

ELEMENTARY TREATISE

ON

ARITHMETIC.

Numeration.

1. A COMPARISON of the different objects, that come within the reach of our senses, soon leads us to perceive, that, in all these objects, there is an attribute, or quality, by which they can be supposed susceptible of increase or diminution; this attribute is *magnitude.* It generally appears in two different forms. Sometimes as a collection of several similar things, or separate parts, and is then designated by the word *number.*

Sometimes it presents itself as a whole, without distinction of parts; it is thus, that we consider the distance between two points, or the length of a line extending from one to the other, as also the outlines and surfaces of bodies, which determine their figure and *extent,* and finally this *extent* itself.

The proper characteristic of this last kind of magnitude, is the connexion or union of the parts, or their *continuity;* whilst in number we consider how many parts there are; a circumstance to which the word quantity at first had relation, though afterwards it was applied to magnitude in general, magnitude considered as a whole being called *continued quantity,* to distinguish it from number, which is called *discrete,* or *discontinued, quantity.*

2. All that relates to magnitude is the object of *mathematics;* numbers, in particular, are the object of *arithmetic.*

Continued magnitude belongs to *geometry,* which treats of the properties presented by the forms of bodies, considered with regard to their *extent.*

3. Number, being a collection of many similar things, or many

distinct parts, supposes the existence of one of these things, or parts, taken as a term of comparison, and this is called *unity.*

The most natural mode of forming numbers is, to begin with joining one unity to another, then, to this sum another; and continuing in this manner, we obtain collections of units, which are expressed by particular names; the whole of these names, which varies in different languages, composes the *spoken numeration.*

4. As there are no limits to the extension of numbers, since however great a number may be, it is always possible to add an unit to it, we may easily conceive that there is an infinity of different numbers, and, consequently, that it would be impossible to express them in any language whatever, by names, that should be independent of each other.

Hence have arisen nomenclatures, in which the object has been, by the combinations of a small number of words, subject to regular forms, and therefore easily remembered, to give a great number of distinct expressions.

Those, which are in use in the [English language,] with few exceptions, are derived from the names assigned to the nine first numbers and those afterwards given to the collections of *ten,* a *hundred,* and a *thousand units.*

The units are expressed by

one, two, three, four, five, six, seven, eight, nine.

The collections of ten units, or *tens,* by

ten, twenty, thirty, forty, fifty, sixty, seventy, eighty, ninety.

The collections of ten tens, or *hundreds,* are expressed by names borrowed from the units; thus we say,

hundred, two hundred, three hundred, ... nine hundred.

The collections of ten hundreds, or *thousands,* receive their denominations from the nine first numbers and from the collections of tens and hundreds; thus we say

thousand, two thousand . . . . nine thousand,

ten thousand, twenty thousand, &c.

hundred thousand, two hundred thousand, &c.

The collections of ten hundred thousands, or of thousands of thousands, take the name of *millions,* and are distinguished, like the collections of thousands.

The collections of ten hundreds of millions, or of thousands of millions, are called *billions*, and are distinguished, like the collections of millions.†

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† The idea of number is the latest and must difficult to form. Before the mind can arrive at such an abstract conception, it must be familiar with that process of classification, by which we successively remount from individuals to species, from species to genera, and from genera to orders. The savage is lost in his attempts at numeration, and significantly expresses his inability to proceed by holding up his expanded fingers, or pointing to the hairs of his head.

Nature has furnished the great and universal standard for computation in the fingers of the hand. All nations have accordingly reckoned by *fives*; and some barbarous tribes have scarcely advanced any further. After the fingers of one hand had been counted once, it was a second and perhaps a distant step to proceed to those of the other. The primitive words expressing numbers did not probably exceed five. To denote *six, seven, eight* and *nine*, the North American Indians repeat the five with the successive addition of one, two, three, and four; could we safely trace the descent and affinity of the abbreviated terms denoting the numbers from five to ten, it seems highly probable, that we should discover a similar process to have taken place in the formation of the most refined languages.

The ten digits of both hands being reckoned up, it then became necessary to repeat the operation. Such is the foundation of our decimal scale of arithmetic. Language still betrays by its structure the original mode of preceding. To express the numbers beyond ten, the Laplanders combine an ordinal, with a cardinal digit. Thus, eleven, twelve, &c. they denominate *second* ten and one, *second* ten and two, &c. and in like manner they call twenty one, twenty two, &c. *third* ten and one, *third* ten and two, &c. Our term *eleven* is supposed to be derived from *ein* or *one*, and *liben, to remain*, and to signify *one, leave* or *set aside ten. Twelve* is of the like derivation and means *two, laying aside the ten.* The same idea is suggested by our termination *ty* in the words *twenty, thirty*, &c. This syllable altogether distinct from *ten* is derived from *ziehen* to *draw*, and the meaning of *twenty* is, strictly speaking, *two drawings*, that is, the hands have been twice closed and the fingers counted over.

After ten was firmly established, as the standard of numeration, it

Each of the names just mentioned is considered as forming a unit of an order more elevated according as it is removed from simple unit. The names *ten* and *hundred* are continually repeated and we have no occasion for new names, such as *thousand, million, billion,* except at every fourth order. The same law being observed, to billions succeed *trillions, quadrillions, quintillions,* &c. each, like billions, having its tens and hundreds.

Numbers expressed in this manner, when more than one word enters into the enunciation of them, are separated into their respective orders of units, mentioned above; for instance, the number expressed by *five hundred thousand three hundred and two,* is separated into three parts, viz. *five hundreds of thousands, three hundreds of simple units, and two of these units.*

5. The length of the expression, written in words, when the numbers were large, occasioned the invention of characters, exclusively adapted to a shorter representation, and hence originated the art of expressing numbers in writing by these characters called *figures,* or *written numeration.*

The laws of the written numeration, now used, are very analogous to those of the spoken numeration. In it the nine first numbers are each represented by a particular character, viz.

1	2	3	4	5	6	7	8	9
one,	two,	three,	four,	five,	six,	seven,	eight,	nine.

When a number consists of tens and units, the characters representing the number of each are written in order from left to right, beginning with the tens. The number forty-seven, for instance, is written 47; the first figure on the left, 4, denotes the four tens, and consequently a value ten times greater than it would have standing alone; while the figure 7, placed on the

---

seemed the most easy and consistent to proceed by the same repeated composition. Both hands being closed ten times would carry the reckoning up to a *hundred.* This word, originally *hund,* is of uncertain derivation; but the term thousand which occurs at the next stage of the progress, or the hundred added ten times is clearly traced out, being only a contraction of *duis hund,* or *twice hundred,* that is, the *repetition,* or *collection of hundreds.* See Edinburgh Review, vol. XVIII. art. VII.

right, expressing seven units, possesses only its original value.

In the number thirty-three, which is written 33, we see the figure 3 repeated, but each time with a different value; the value of the 3 on the left is ten times greater than the value of that on the right.

This is the fundamental law of our written numeration, that *a removal, of one place, towards the left increases the value of a figure ten times.*

If it were required to express fifty, or five tens, as there are no units in this number, there would be nothing to write but the figure 5, and consequently it would be necessary to show, by some particular mark, that in the expression of this number, the figure ought to occupy the first place on the left. To do this we place on the right the character 0, *cipher* or *nought,* which of itself has no value, and serves only to fill the place of the units, which are wanting in the enunciation of the proposed number.

6. Thus with ten characters, by means of the rule before laid down concerning the value which figures assume, according to the places they occupy, we can express all possible numbers.

With two figures only, we can write all, as far as to nine tens and nine units, making 99, or ninety nine. After this comes the hundred, which is expressed by the figure 1, put one place farther towards the left, than it would be, if used to express tens only; and to denote this place, two ciphers are placed on the right, making 100.

The units and tens, afterwards added to form numbers greater than 100, take their proper places; thus a hundred and one will be written in figures 101; a hundred and eleven, 111. Here the same figure is three times repeated, and with a different value each time; in the first place on the right it expresses an unit, in the second, a ten, in the third, a hundred. It is the same with the number 222, 333, 444, &c. Thus, in consequence of the rule laid down before when speaking of units and tens, *the same figure expresses units ten times greater, in proportion as it is removed from right to left, and by a simple change of place, acquires the power of representing successively, all the different collections of units, which can enter into the expression of a number.*

7. A number dictated, or enunciated, is written then, by placing one after the other, beginning at the left, the figures which express the number of units of each collection; but it is necessary to keep in mind the order in which the collections succeed each other, that no one may be omitted, and to put ciphers in the room of those, which are wanting in the enunciation of the number to be written. If, for example, the number were *three hundred and twenty four thousand, nine hundred and four,* we should put 3 for the hundreds of thousands, 2 for the twenty thousand, or the two tens of thousands, 4 for the thousands, 9 for the hundreds; and as the tens come immediately after the hundreds, and are wanting in the given number, we should put a cipher in the room of them, and then write the figure 4 for the units; we should thus have 324904.

In the same way, writing ciphers in the place of tens of thousands, thousands and tens, which are wanting in the number five hundred thousand three hundred and two, we should have 500302.

8. When a number is written in figures, in enunciating it, or expressing it in language, it is necessary to substitute for each of the figures the word which it represents, and then to mention the collection of units, to which it belongs according to the place it occupies. The following example will illustrate this;

2	4,	8	9	7,	3	2	1,	5	8	0,	3	4	6,
Tens of Trillions	TRILLIONS	Hundreds of Billions	Tens of Billions	BILLIONS	Hundreds of Millions	Tens of Millions	MILLIONS	Hundreds of Thousands	Tens of Thousands	THOUSANDS	Hundreds	Tens	UNITS

The figures of this number are divided by commas, into portions of three figures each, beginning at the right; but the last division on the left, which in the present instance has but two figures, may sometimes have but one. Each of these divisions corresponds to the collections designated by the words *unit, thousand,* *million, billion, trillion,* and their figures express successively the units, tens and hundreds of each. *Consequently, the expression of the whole number given is made in words, by reading each division of figures as if it stood alone, and adding, after its units, the name of their place.*

The above example is read, *twenty four trillions, eight hundred and ninety seven billions, three hundred and twenty one millions, five hundred and eighty thousand, three hundred and forty six units.*

9. Numbers admit of being considered in two ways; one is, when no particular denomination is mentioned, to which their units belong, and they are then called *abstract numbers;* the other when the denomination of their units is specified, as when we say, two men, five years, three hours, &c. these are called *concrete numbers.*

It is evident, that the formation of numbers, by the successive union of units, is independent of the nature of these units, and that this must also be the case with the properties resulting from this formation; by which properties we are enabled to compound and decompound numbers, which is called *calculation.* We shall now explain the principal rules for the calculation of numbers, without regard to the nature of their units.

Addition.

10. This operation, which has for its object the uniting of several numbers in one, is only an abbreviation of the formation of numbers by the successive union of units. If, for instance, it were required to add five to seven, it would be necessary, in the series of the names of numbers, *one, two, three, four, five, six, seven,* &c. to ascend five places above seven, and we should then come to the word *twelve,* which is consequently the amount of seven units added to five. It is upon this process that the addition of all small numbers depends, the results of which are committed to memory; its immediate application to larger numbers would be impossible, but in this case, we suppose these numbers divided into the different collections of units contained in them, and we may add together those of the same name. For instance, to add 27 to 32, we add the 7 units of the first number to the 2 of the second, making 9; then the 2 tens of the first with

the 3 of the second, making 5 tens. The two results, taken together, form a total of 5 tens and 9 units or 59, which is the sum of the numbers proposed.

What is here said, applies to all numbers however large, that are to be added together, but it is necessary to observe that the partial sums, resulting from the addition of two numbers, each expressed by a single figure, often contain tens, or units of the next higher collection, and these ought consequently to be joined to their proper collection.

In the addition of the numbers 49 and 78, the sum of the units 9 and 8 is 17, of which we should reserve 10, or ten, to be added to the sum of the tens in the given numbers; next we say that 4 and 7 make 11, and joining to this the ten we reserved, we have 12 for the number of tens contained in the sum of the given numbers; which sum, therefore, contains 1 hundred, 2 tens and 7 units, that is 127.

11. By proceeding on these principles, a method has been devisde of placing numbers, that are to be added, which facilitates the uniting of their collections of units, and a rule has been formed which the following example will illustrate.

Let the numbers be 527, 2519, 9812, 73 and 8; in order to add them together, we begin by writing them under each other, placing the units of the same order in the same column; then we draw a line to separate them from the result, which is to be written underneath it.

	527
	2519
	9812
	73
	8
Sum	12939

We at first find the sum of the numbers contained in the column of units to be 29, we write down only the nine units, and reserve the 2 tens, to be joined to those which are contained in the next column, which, thus increased, contains 13 units of its own order; we write down here only the three units, and carry the ten to the next column. Proceeding with this column as with the others, we find its sum to be 19; we write down the 9 units and carry the ten to the next column, the sum of which we then find to be 12; we write down the 2 units under this column and place the ten on the left of it; that is, we write down the sum of this column, as it is found.

By this means we obtain 12939 for the sum of the given numbers.

12. The rule for performing this operation may be given thus,

Write the numbers to be added, under each other, so that all the units of the same kind may stand in the same column, and draw a line under them.

Beginning at the right, add up successively the numbers in each column; if the sum does not exceed 9, write it beneath its column, as it is found; if it contains one or more tens, carry them to the next column; lastly, under the last column write the whole of its sum†.

Examples for practice.

Add together 8635, 2194, 7421, 5063, 2196 and 1225. *Ans.* 26734.

Add together 84371, 6250, 10, 3842 and 631. *Ans.* 95104.

Add together 3004, 523, 8710, 6345 and 784. *Ans.* 19366.

Add together 7861, 345, 8023. *Ans.* 16229.

Add together 66947, 46742 and 132684. *Ans.* 246373.

Subtraction.

13. After having learned to compose a number by the addition of several others, the first question, that presents itself, is, how to take one number from another that is greater, or which amounts to the same thing, to separate this last into two parts, one of which shall be the given number. If, for instance, we have the

† The best method of proving addition is by means of subtraction The learner may however, in general, satisfy himself of the correctness of his work by beginning at the top of each column and adding down, or by separating the upper line of figures and adding up the rest and then adding this sum to the upper line.

number 9, and we wish to take 4 from it, we should, by doing this, separate it into two parts, which by addition would be the same again.

To take one number from another, when they are not large, it is necessary to pursue a course opposite to that prescribed, in the beginning of article 10, for finding their sum; that is, in the series of the names of numbers, we ought to begin from the greatest of the numbers in question, and descend as many places as there are units in the smallest, and we shall come to the name given to the difference required. Thus, in descending four places below the number *nine*, we come to *five*, which expresses the number that must be added to 4 to make 9, or which shows how much 9 is greater than 4.

In this last point of view, 5 is the *excess* of 9 above 4. If we only wished to show the inequality of the numbers 9 and 4, without fixing our attention on the order of their values, we should say that their *difference* was 5. Lastly, if we were to go through the operation of taking 4 from 9, we should say that the *remainder* is 5. Thus we see that, although the words, *excess*, *remainder*, and *difference*, are synonymous, each answers to a particular manner of considering the separation of the number 9 into the parts 4 and 5, which operation is always designated by the name *subtraction.*

14. When the numbers are large, the subtraction is performed, part at a time, by taking successively from the units of each order in the greatest number, the corresponding units in the least. That this may be done conveniently, the numbers are placed as 9587 and 345 in the following example;

	9587
	345
Remainder	9242

and under each column is placed the excess of the upper number, in that column, over the lower, thus;

5, taken from 7, leaves 2,
4, taken trom 8, leaves 4,
3, taken from 5, leaves 2,

and writing afterwards the figure 9, from which there is nothing to be taken; the remainder, 9242, shows how much 9587 is greater than 345.

That the process here pursued gives a true result is indisputable, because in taking from the greatest of the two numbers all the parts of the least, we evidently take from it the whole of the least.

15. The application of this process requires particular attention, when some of the orders of units in the upper number are greater than the corresponding orders in the lower.

If, for instance, 397 is to be taken from 524.

	524
	397
Remainder	127

In performing this question we cannot at first take the units in the lower number from those in the upper; but the number 524, here represented by 4 units, 2 tens and 5 hundreds, can be expressed in a different manner by decomposing some of its collections of units, and uniting a part with the units of a lower order. Instead of the 2 tens and 4 units which terminate it we can substitute in our minds 1 ten and 14 units, then taking from these units the 7 of the lower number, we get the remainder 7. By this decomposition, the upper number now has but one ten, from which we cannot take the 9 of the lower number, but from the 5 hundred of the upper number we can take 1, to join with the ten that is left, and we shall then have 4 hundreds and 11 tens, taking from these tens the tens of the lower number, 2 will remain. Lastly, taking from the 4 hundreds, that are left in the upper number, the three hundreds of the lower, we obtain the remainder 1, and thus get 127 as the result of the operation.

This manner of working consists, as we see, in borrowing, from the next higher order, an unit, and joining it according to its value to those of the order, on which we are employed, observing to count the upper figure of the order from which it was borrowed one unit less, when we shall have come to it.

16. When any orders of units are wanting in the upper number, that is, when there are ciphers between its figures, it is

necessary to go to the first figure on the left, to borrow the 10 that is wanted. See an example

7002
3495

Remainder 3507.

As we cannot take the 5 units of the lower number from the 2 of the upper, we borrow 10 units from the 7000, denoted by the figure 7, which leaves 6990; joining the 10 we borrowed to the figure 2. the upper number is now decompounded into 6990 and 12; taking from 12 the 5 units of the lower number, we obtain 7 for the units of the remainder.

This first operation has left in the upper number 6990 units or 699 tens instead of the 700 expressed by the three last figures on the left; thus the places of the two ciphers are occupied by 9s and the significant figure on the left is diminished by unity. Continuing the subtraction in the other columns in the same manner, no difficulty occurs, and we find the remainder, as put down in the example.

17. Recapitulating the remarks made in the two preceding articles, the rule to be observed in performing subtraction may be given thus. *Place the less number under the greater, so that their units of the same order may be in the same column, and draw a line under them; beginning at the right take successively each figure of the lower number from the one in the same column of the upper; if this cannot be done, increase the upper figure by ten units, counting the next significant figure, in the upper member, less by unity, and if ciphers come between, regard them as 9s.*

18. For greater convenience, when it is necessary to decrease the upper figure by unity, we can suffer it to retain its value, and add this unit to the corresponding lower figure, which, thus increased, gives as is wanted, a result one less than would arise from the written figures. In the first of the following examples, after having taken 6 units from 14, we count the next figure of the lower number 8, as 9, and so in the others.

Examples.

16844	10378	103034	49812002
9786	2437	69845	18924983
7058		33189	

173425	8037142	2123724	39742107
57632	5067310	1123467	25378421

Method of proving Addition and Subtraction.

19. In performing an operation, according to a process, the correctness of which is established upon fixed principles, we may nevertheless sometimes commit errours in the partial additions and subtractions, the results of which we seek in the memory. To prevent any mistake of this kind, we have recourse to a method, the reverse of the first operation, by which we ascertain whether the results are right; this is called *proving* the operation.

The proof of addition consists in subtracting successively from the sum of the numbers added, all the parts of these numbers, and if the work has been correctly performed, there will be no remainder. We will now show by the example given in article 11, how to perform all these subtractions at once.

	527
	2519
	9812
	73
	8
Sum	12939
	1120

We first add the numbers in the left hand column, which here contains thousands, and subtract the sum 11 from 12, which begins the preceding result, and write underneath the difference 1, produced by what was reserved from the column of hundreds, in performing the addition. The sum of the column of hundreds taken by itself, amounts to but 18, if we take this from the 9 of the first result, increased by borrowing the one thousand, considered as ten hundred, that remains from the column preceding it on the left, the remainder 1, written beneath will show what was reserved from the column of tens. The sum of these last 11, taken from 13, leaves for its remainder 2 tens, the number reserved from the column of units. Joining these 2 tens with the 9 units of the answer, we form the number 29, which ought to be exactly the sum of the column of units, as this column is not affected by any of the others; adding again the numbers in this column, we ought to come to the same result, and consequently, to have no remainder. This is actually the case, as is denoted by the 0 written under the column. The process, just explained, may be given thus; *to prove addition, beginning on the left, add again each of the several columns, subtract the sums respectively from the sums written above them and write down the remainders, which must be joined, each as so many tens to the sum of the next column on the right; if the work be correct there will be no remainder under the last column.*

20. The proof of subtraction is, that *the remainder, added to the least number, exactly gives the greatest.* Thus to ascertain the exactness of the following subtraction,

$$\begin{array}{r} 524 \\ 297 \\ \hline 227 \\ \hline 524 \end{array}$$

we add the remainder to the smallest number, and find the sum, in reality, equal to the greatest.

Multiplication.

21. When the numbers to be added are equal to each other, addition takes the name of *multiplication,* because in this case the sum is composed of one of the numbers repeated as many times as there are numbers to be added. Reciprocally, if we wish to repeat a number several times, we may do it, by adding the number to itself as many times, wanting one, as it is to be repeated. For instance, by the following addition,

16
16
16
16
—
64

the number 16 is repeated four times, and added to itself three times.

To repeat a number twice is to *double* it; 3 times, to *triple* it; 4 times, to *quadruple* it, and so on.

22. Multiplication implies three numbers, namely, that, which is to be repeated, and which is called the *multiplicand;* the number which shows how many times it is to be repeated, which is called the *multiplier;* and lastly, the result of the operation, which is called the *product.* The *multiplicand* and *multiplier,* considered as concurring to form the product, are called *factors* of the *product.* In the example given above, 16 is the *multiplicand,* 4 the *multiplier,* and 64 the *product;* and we see that 4 and 16 are the *factors* of 64.

23. When the multiplicand and multiplier are large numbers, the formation of the product, by the repeated addition of the multiplicand, would be very tedious. In consequence of this, means have been sought of abridging it, by separating it into a certain number of partial operations, easily performed by memory. For instance, the number 16 would be repeated 4 times, by taking separately, the same number of times, the 6 units and the ten, that compose it. It is sufficient then to know the products arising from the multiplication of the units of each order in the multiplicand by the multiplier, when the multiplier consists of a single figure, and this amounts, for all cases that can occur, to finding the products of each one of the 9 first numbers by every other of these numbers.

24. These products are contained in the following table, attributed to Pythagoras.

TABLE OF PYTHAGORAS.

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

25. To form this table, the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, are written first on the same line. Each one of these numbers is then added to itself and the sum written in the second line, which thus contains each number of the first doubled, or the product of each number by 2. Each number of the second line is then added to the number over it in the first, and their sums are written in the third line, which thus contains the triple of each number in the first, or their products be 3. By adding the numbers of the third line to those of the first, a fourth is formed, containing the quadruple of each number of the first, or their products by 4; and so on, to the ninth line, which contains the products of each number of the first line by 9.

It may not be amiss to remark, that the different products of any number whatever by the numbers 2, 3, 4, 5, &c. are called *multiples* of that number; thus 6, 9, 12, 15, &c. are multiples of 3.

26. When the formation of this table is well understood, the mode of using it may be easily conceived. If, for instance, the product of 7 by 5 were required; looking to the fifth line, which contains the different products of the 9 first numbers by 5, we should take the one directly under the 7, which is 35; the same

method should be pursued in every other instance, and *the product will always be found in the line of the multiplier and under the multiplicand.*

27. If we seek in the table of Pythagoras the product of 5 by 7, we shall find, as before, 35, although in this case 5 is the mulplicand, and 7 the multiplier. This remark is applicable to each product in the table, and *it is possible, in any multiplication, to reverse the order of the factors; that is, to make the multiplicand the multiplier, and the multiplier the multiplicand.*

As the table of Pythagoras contains but a limited number of products, it would not be sufficient to verify the above conclusion, by this table; for a doubt might arise respecting it in the case of greater products, the number of which is unlimited; there is but one method independent of the particular value of the multiplicand and multiplier, of showing that there is no exception to this remark. This is one well calculated for the purpose, as it gives a good illustration of the manner, in which the product of two numbers is formed. To make it more easily understood, we will apply it first to the factors 5 and 3.

If we write the figure 1, 5 times on one line, and place two similar lines underneath the first, in this manner,

1, 1, 1, 1, 1,
1, 1, 1, 1, 1,
1, 1, 1, 1, 1,

the whole number of 1s will consist of as many times 5 as there are lines, that is, 3 times 5; but, by the disposition of these lines, the figures are ranged in columns, containing 3 each. Counting them in this manner, we find as many times 3 units as there are columns, or 5 times 3 units, and as the product does not depend on the manner of counting, it follows that 3 times 5, and 5 times 3 give the same product. It is easy to extend this reasoning to any numbers, if we conceive each line to contain as many units as there are in the multiplicand, and the number of lines, placed one under the other, to be equal to the multiplier. In counting the product by lines, it arises from the multiplicand repeated as many times as there are units in the multiplier; but the assemblage of figures written, presents as many columns as there are units in a line, and each column contains as many units as there are lines; if then, we choose to count by columns, the number of lines, or the multiplier, will be repeated as many times as there are units in a line, that is, in the multiplicand. We may therefore, in finding the product of any two numbers, take either of them at pleasure, for the multiplier.

28. The reasoning just given to prove the truth of the preceding proposition, is the demonstration of it, and it may be remarked, that the essential distinction of pure mathematics is, that no proposition, or process, is admitted, which is not the necessary consequence of the primary notions, on which it is founded, or the truth of which is not generally established by reasoning independent of particular examples, which can never constitute a proof, but serve only to facilitate the reader's understanding the reasoning, or the practice of the rules.

29. Knowing all the products given by the nine first numbers, combined with each other, we can, according to the remark in article 23, multiply any number by a number consisting of a single figure, by forming successively the product of each order of units in the multiplicand, by the multiplier; the work is as follows;

$$\begin{array}{r} 526 \\ 7 \\ \hline 3682 \end{array}$$

The product of the units of the multiplicand, 6, by the multiplier, 7, being 42, we write down only the 2 units, reserving the 4 tens to be joined with those, that will be found in the next higher place.

The product of the tens of the multiplicand, 2, by the multiplier, 7, is 14, and adding the 4 tens we reserved, we make them 18, of which number we write only the units, and reserve the ten for the next operation.

The product of the hundreds of the multiplicand, 5, by the multiplier, 7, is 35, when increased by the 1 we reserved, it becomes 36, the whole of which is written, because there are no more figures in the multiplicand.

30. This process may be given thus; *To multiply a number*

of several figures by a single figure, place the multiplier under the units of the multiplicand, and draw a line beneath, to separate them from the product. Beginning at the right, multiply successively, by the multiplier, the units of each order in the multiplicand, and write the whole product of each, when it does not exceed 9; but if it contains tens, reserve them to be added to the next product. Continue thus to the last figure of the multiplicand, on the left, the whole result of which must be written down.

Examples. 243 by 6. Ans. 1458. 8943 by 9. Ans. 80487.

It is evident that, when the multiplicand is terminated by 0s, the operation can commence only with its first significant figure; but to give the product its proper value, it is necessary to put, on the right of it, as many 0s as there are in the multiplicand. As for the 0s, which may occur between the figures of the multiplicand, they give no product, and a 0 must be written down when no number has been reserved from the preceding product, as is shown by the following examples;

956	8200	7012	80970
6	9	5	4
5736	73800	35060	323880

Multiply

730 by 3. *Ans.* 2190. 8104 by 4. *Ans.* 32416.

20508 by 5. *Ans.* 102540. 360500 by 6. *Ans.* 2163000.

297000 by 7. *Ans.* 2079000. 9097030 by 9. *Ans.* 81873270.

31. The most simple number, expressed by several figures, being 10, 100, 1000, &c. it seems necessary to inquire how we can multiply any number by one of these. Now if we recollect the principle mentioned in article 6, by which the same figure is increased in value 10 times, by every remove towards the left, we shall soon perceive, that, to multiply any number by 10, we must make each of its orders of units ten times greater; that is, we must change its units into tens, its tens into hundreds, and so on, and that this is effected by placing a 0 on the right of the number proposed, because then all its significant figures will be advanced one place towards the left.

For the same reason, to multiply any number by 100, we should place two ciphers on the right; for, since it becomes ten times greater by the first cipher, the second will make it ten times greater still, and consequently it will be 10 times 10, or 100 times, greater than it was at first.

Continuing this reasoning, it will be perceived that, according to our system of numeration, a number is multiplied by 10, 100, 1000, &c. by writing on the right of the multiplicand as many ciphers, as there are on the right of the unit in the multiplier.

32. When the significant figure of the multiplier differs from unity, as, for instance, when it is required to multiply by 30, or 300, or 3000, which are only 10 times 3, or 100 times 3, or 1000 times 3, &c. the operation is made to consist of two parts, we at first multiply by the significant figure, 3, according to the rule in article 30, and then multiply the product by 10, 100, or 1000, &c. (as was stated in the preceding article,) by writing one, two, three, &c. ciphers on the right of this product.

Let it be required, for instance, to multiply 764 by 300.

$$\begin{array}{r} 764 \\ \underline{300} \\ 229200 \end{array}$$

The four significant figures of this product result from the multiplication of 764 by 3, and are placed two places towards the left to admit the two ciphers, which terminate the multiplier.

In general, *when the multiplier is terminated by a number of ciphers, first multiply the multiplicand by the significant figure of the multiplier, and place, after the product, as many ciphers as there are in the multiplier.*

Examples.

Multiply

35012 by 100. *Ans.* 3501200. 638427 by 500. *Ans.* 319213500.
2107900 by 70. *Ans.* 147553000. 9120400 by 90. *Ans.* 820836000.

33. The preceding rules apply to the case, in which the multiplier is any number whatever, by considering separately each of the collections of units of which it is composed. To multiply, for instance, 793 by 345, or which is the same thing, to repeat 793, 345 times, is to take 793, 5 times, added to 40 times, added to

300 times, and the operation to be performed is resolved into 3 others, in each of which the multipliers, 5, 40, and 300, have but one significant figure.

To add the result of these three operations easily, the calculation is disposed thus;

```
   793
   345
  ----
  3965
 31720
237900
------
273585
```

The multiplicand is multiplied successively by the units, tens, hundreds, &c. of the multiplier, observing to place a cipher on the right of the partial product, given by the tens in the multiplier, and two on the right of the product given by hundreds, which advances the first of these products one place towards the left, and the second, two. The three partial products are then added together, to obtain the total product of the given numbers.

As the ciphers, placed at the end of these partial products, are of no value in the addition, we may dispense with writing them, provided we take care to put in its proper place, the first figure of the product given by each significant figure of the multiplier; that is, to put in the place of tens, the first figure of the product given by the tens in the multiplier; in the place of hundreds the first figure of the product given by the hundreds in the multiplier, and so on.

34. According to what has been said, the rule is as follows. *To multiply any two numbers, one by the other, form successively (according to the rule in article 30,) the products of the multiplicand, by the different orders of units in the multiplier; observing to place the first figure of each partial product under the units of the same order with the figure of the multiplier, by which the product is given; and then add together all the partial products.*

35. When the multiplicand is terminated by ciphers, they may at first be neglected, and all the partial multiplications begin with the first significant figure of the multiplicand; but after-

wards, to put in their proper rank the figures of the total product, as many ciphers, as there are in the multiplicand, must be written on the right of this product.

If the multiplier is terminated by ciphers, we may, according to the remark in article 31, neglect these also, provided we write an equal number on the right of the product.

Hence it results that, *when both multiplicand and multiplier are terminated by ciphers, these ciphers may at first be neglected, and after the other figures of the product are obtained, the same number may be written on the right of the product.*

When there are ciphers between the significant figures of the multiplier, as they give no product, they may be passed over, observing to put in its proper place, the unit of the product given by the figure on the left of these ciphers.

Examples.

300	526	Multiply 9648 by 5137. *Ans.* 49561776.
40	307	7854 by 350. *Ans.* 27489000.
———	———	17204774 by 125. *Ans.* 2150596750.
12000	3682	62500 by 520. *Ans.* 32500000.
	157800	25980762 by 20. *Ans.* 10392304800.
	———	
	161482	

Division.

36. The product of two numbers being formed by repeating one of these numbers as many times as there are units in the other, we can, from the product, find one of the factors, by ascertaining how many times it contains the other; subtraction alone is necessary for this. Thus, if it be required to ascertain the number of times 64 contains 16, we need only subtract 16 from 64 as many times as it can be done; and since, after 4 subtractions, nothing is left, we conclude, that 16 is contained 4 times in 64. This manner of decomposing one number by another, in order to know how many times the last is contained in the first, is called *division*, because it serves to divide, or portion out, a given number into equal parts, of which the number or value is given.

If, for instance, it were required to divide 64 into 4 equal parts; to find the value of these parts, it would be necessary to ascertain the number, that is contained 4 times in 64, and consequently to regard 64 as a product, having for its factors, 4 and one of the required parts, which is here 16.

If it were asked how many parts, of 16 each, 64 is composed of, it would be necessary, in order to ascertain the number of these parts, to find how many times 64 contains 16, and consequently, 64 must be regarded as a product, of which one of the factors is 16, and the other the number sought, which is 4.

Whatever then may be the object in view, *division consists in finding one of the factors of a given product, when the other is known.*

37. The number to be divided is called the *dividend*, the factor, that is known, and by which we must divide, is called the *divisor*, the factor found by the division is called the *quotient*, and always shows how times the divisor is contained in the dividend.

It follows then, from what has been said, *that the divisor multiplied by the quotient, ought to reproduce the dividend.*

38. When the dividend can contain the divisor a great many times, it would be inconvenient in practice to make use of repeated subtraction for finding the quotient; it then becomes necessary to have recourse to an abbreviation analogous to that which is given for multiplication. If the dividend is not ten times larger than the divisor, which may be easily perceived by the inspection of the numbers, and if the divisor consists of only one figure, the quotient may be found by the table of Pythagoras, since that contains all the products of factors, that consist of only one figure each. If it were asked, for instance, how many times 8 is contained in 56, it would be necessary to go down the 8th column, to the line in which 56 is found; the figure 7, at the beginning of this line, shows the second factor of the number 56, or how many times 8 is contained in this number.

We see by the same table, that there are numbers, which cannot be exactly divided by others. For instance, as the seventh line, which contains all the multiples of 7, has not 40 in it, it follows that 40 is not divisible by 7; but as it comes between 35 and 42, we see that the greatest multiple of 7, it can contain, is 35, the factors of which are 5 and 7. By means of this elementary information, and the considerations, which will now be offered, any division whatever may be performed.

39. Let it be required, for example, to divide 1656 by 3; this question may be changed into another form, namely; *To find such a number, that multiplying its units, tens, hundreds, &c. by 3, the product of these units, tens, hundreds, &c. may be the dividend,* 1656.

It is plain, that this number will not have units of a higher order than thousands, for, if it had tens of thousands, there would be tens of thousands in the product, which is not the case. Neither can it have units of as high an order as thousands, for if it had but one of this order, the product would contain at least 3, which is not the case. It appears then, that the thousand in the dividend is a number reserved, when the hundreds of the quotient were multiplied by 3, the divisor.

This premised, the figure occupying the place of hundreds, in the required quotient, ought to be such, that, when multiplied by 3, its product may be 16, or the greatest multiple of 3, less than 16. This restriction is necessary, on account of the reserved numbers, which the other figures of the quotient may furnish, when multiplied by the divisor, and which should be united to the product of the hundreds.

The number, which fulfils this condition is 5; but 5 hundreds, multiplied by 3, gives 15 hundreds, and the dividend, 1656, contains 16 hundreds; the difference, 1 hundred, must have come then from the reserved number, arising from the multiplication of the other figures of the quotient by the divisor. If we now subtract the partial product 15 hundreds, or 1500, from the total product 1656, the remainder 156, will contain the product of the units and tens of the quotient by the divisor, and the question will be reduced to finding a number, which, multiplied by 3, gives 156, a question similar to that, which presented itself above. Thus when the first figure of the quotient shall have been found in this last question, as it was in the first, let it be multiplied by the divisor, then subtracting this partial product from the whole product, the result will be a new dividend, which may be treated in the same manner as the preceding, and so on, until the original dividend is exhausted.

40. The operation just described is disposed of thus;

```
dividend 1656 |   3 divisor
         15   |  552 quotient
        ------|
          15
          15
          --
           06
            6
            -
            0
```

The dividend and divisor are separated by a line, and another line is drawn under the divisor, to mark the place of the quotient. This being done, we take on the left of the dividend the part 16, capable of containing the divisor, 3, and dividing it by this number, we get 5 for the first figure of the quotient on the left; then taking the product of the divisor by the number just found, and subtracting it from 16, the partial dividend, we write underneath, the remainder, 1, by the side of which we bring down the 5 tens of the dividend. Considering the number, as it now stands, a second partial dividend, we divide it also by the divisor 3, and obtain 5 for the second figure of the quotient; we then take the product of this number by the divisor, and subtracting it from the partial dividend, get 0 for the remainder. We then bring down the last figure of the dividend, 6, and divide this third partial dividend by the divisor, 3, and get 2 for the last figure of the quotient.

41. It is manifest that, if we find a partial dividend, which cannot contain the divisor, it must be because the quotient has no units of the order of that dividend, and that those which it contains arise from the products of the divisor by the units of the lower orders in the quotient; it is necessary therefore, whenever this is the case, to put a 0 in the quotient, to occupy the place of the order of units that is wanting.

For instance, let 1535 be divided by 5.

```
1535 | 5
15   |----
-----| 307
 035
  35
 ---
  00
```

The division of the 15 hundreds of the dividend by the divisor, leaving no remainder, the 3 tens, which form the second partial dividend, do not contain the divisor. Hence it appears, that the quotient ought to have no tens; consequently this place must be filled with a cipher, in order to give to the first figure of the quotient the value, it ought to have, compared with the others; then bringing down the last figure of the dividend, we form a third partial dividend, which, divided by 5, gives 7 for the units of the quotient, the whole of which is now 307.

42. The considerations, presented in article 40, apply equally to the case, in which the divisor consists of any number of figures.

If, for instance, it were required to divide 57981 by 251, it would easily be seen, that the quotient can have no figures of a higher order than hundreds, because, if it had thousands, the dividend would contain hundreds of thousands, which is not the case; further, the number of hundreds should be such, that, multiplied by 251, the product would be 579, or the multiple of 251 next less than 579; this restriction is necessary on account of the reserved numbers which may have been furnished by the multiplication of the other figures of the quotient by the divisor. The number, which answers to this condition, is 2; but 2 hundreds, multiplied by 251, give 502 hundreds, and the divisor contains 579; the difference, 77 hundreds, arises then from the reserved numbers resulting from the multiplication of the units and tens of the quotient, by the divisor.

If we now subtract the partial product, 502 hundreds, or 50200, from the total product, 57981, the remainder 7781, will contain the products of the units and tens of the quotient by the divisor, and the operation will be reduced to finding a number, which, multiplied by 251, will give for a product 7781.

Thus, when the first figure of the quotient shall have been determined, it must be multiplied by the divisor, the product being subtracted from the whole dividend, a new dividend will be the result, which must be operated upon like the preceding; and so on, till the whole dividend is exhausted.

It is always necessary, for obtaining the first figure of the quotient, to separate, on the left of the dividend, so many figures, as, considered as simple units, will contain the divisor, and admit of this partial division.

43. Disposing of the operation as before, the calculation, just explained, is performed in the following order;

```
57981 | 251
502   |----
------| 231
 778
 753
 ---
  251
  251
  ---
  000
```

The 3 first figures, on the left of the dividend are taken to form the partial dividend; they are divided by the divisor, and the number 2, thence resulting, is written in the quotient; the divisor is then multiplied by this number, and the product, 502, is written under the partial dividend, 579. Subtraction being performed, the 8 tens of the dividend are brought down to the side of the remainder, 77; this new partial dividend is then divided by the divisor, and 3 is obtained for the second figure of the quotient; the divisor is multiplied by this, the product subtracted from the corresponding partial dividend, and to the remainder, 25, is brought down the last figure of the dividend, 1; this last partial dividend, 251, being equal to the divisor, gives 1 for the units of the quotient.

44. When the divisor contains many figures, some difficulty may be found in ascertaining how many times it is contained in

the partial dividends. The following example is designed to show how it may be known.

```
423405 | 485
3880   |-----
-------| 873
 3540
 3395
 ----
  1455
  1455
  ----
  0000
```

It is necessary at first to take four figures on the left of the dividend, to form a number which will contain the divisor; and then it cannot be immediately perceived how many times 485 is contained in 4234. To aid us in this inquiry, we shall observe, that this divisor is between 400 and 500; and if it were exactly one or the other of these numbers, the question would be reduced to finding how many times 4 hundred or 5 hundred is contained in the 42 hundreds of the number 4234, or, which amounts to the same thing, how many times 4 or 5 is contained in 42. For the first of these numbers we get 10, and for the second 8, the quotient must now be sought between these two. We see at first that we cannot employ 10, because this would imply, that the order of units in the dividend above hundreds, contained the divisor, which is not the case. It only remains then, to try which of the two numbers 9 or 8, used as the multiplier of 485, gives a product that can be subtracted from 4234, and 8 is found to be the one. Subtracting from the partial dividend the product of the divisor multiplied by 8, we get, for the remainder, 354; bringing down then the 0 tens in the dividend, we form a second partial dividend, on which we operate as on the preceding; and so with the others.

45. The recapitulation of the preceding articles gives us this rule, *To divide one number by another, place the divisor on the right of the dividend, separate them by a line, and draw another line under the divisor, to make the place for the quotient. Take, on the left of the dividend, as many figures as are necessary to contain*

the divisor; find how many times the number, expressed by the first figure of the divisor, is contained in that, represented by the first, or two first, figures of the partial dividend; multiply this quotient, which is only an approximation, by the divisor, and, if the product is greater than the partial dividend, take units from the quotient continually, till it will give a product that can be subtracted from the partial dividend; subtract this product, and if the remainder be greater than the dividend, it will be a proof that the quotient has been too much diminished; and, consequently, it must be increased. By the side of the remainder bring down the next figure of the dividend, and find, as before, how many times this partial dividend contains the divisor; continue thus, until all the figures of the given dividend are brought down. When a partial dividend occurs, which does not contain the divisor, it is necessary, before bringing down another figure of the dividend, to put a cipher in the quotient.

46. The operations required in division may be made to occupy a less space, by performing mentally the subtraction of the products given by the divisor and each figure of the quotient, as is exhibited in the following example;

1755	39
195	45
000	

After having found that the first partial dividend contains 4 times the divisor, 39, we multiply at first the 9 units by 4, which gives 36; and, in order to subtract this product from the partial dividend, we add to the 5 units in the dividend, 4 tens, making their sum 45, from which taking 36, 9 remain. We then reserve 4 tens to join them, in the mind, to 12, the product of the quotient by the tens in the divisor, making the sum 16; in taking this sum from 17, we take away the 4 tens, with which we had augmented the units of the dividend, in order to perform the preceding subtraction. We then operate in the same manner on the second partial dividend, 195, saying; 9 times 5 make 45, taken from 45, nought remains, then 5 times 3 make 15, and 4 tens, reserved, make 19, taken from 19, nought remains.

We see sufficiently by this in what manner we are to perform any other example, however complicated.

47. Division is also abbreviated when the dividend and divisor are terminated by ciphers, because we can strike out, from the end of each, as many ciphers as are contained in the one that has the least number.

If, for instance, 84000 were to be divided by 400, these numbers may be reduced to 840 and 4, and the quotient would not be altered; for we should only have to change the name of the units, since, instead of 84000, or 840 hundreds, and 400, or 4 hundreds, we should have 840 units and 4 units, and the quotient of the numbers 840 and 4 is always the same, whatever may be the denomination of their units.

It may also be remarked that, in striking out two ciphers at the end of the given numbers, they have been, at the same time, both of them divided by 100; for it follows from article 31, that in striking out 1, 2 or 3 ciphers on the right of any number, the number is divided by 10, or 100, or 1000, &c.

Examples in Division.

144	3	16512	344	3049164	6274
24	48	2752	48	53956	486
00		0000		37644	
				00000	

Divide	49561776	by 5137.	*Ans.* 9648.
	27489000	by 350.	*Ans.* 7854.
	2150596750	by 125.	*Ans.* 17204774.
	32500000	by 520.	*Ans.* 62500.
	10392304800	by 20.	*Ans.* 25980762.

48. Division and multiplication mutually prove each other, like subtraction and addition, for according to the definition of division, (36), we ought, by dividing the product by one of the factors, to find the other; and multiplying the divisor by the quotient we ought to reproduce the dividend (37).

Fractions.

49. Division cannot always be exactly performed, because any number whatever of units taken a certain number of times, does not always compose any other number whatever. Exam-

ples of this have already been seen in the table of Pythagoras, which contains only the product of the 9 first numbers, multiplied two and two, but does not contain all the numbers between 1 and 81, the first and last numbers in it. The method hitherto given shows then, only how to find the greatest multiple of the divisor, that can be contained in the dividend.

If we divide 239 by 8, according to the rule in article 46.

239	8
79	29
7	

we have, for the last partial dividend, the number 79, which does not contain 8 exactly, but which, falling between the two numbers, 72 and 80, one of which contains the divisor, 8, nine times, and the other ten, shows us that the last part of the quotient is greater than 9, and less than 10, and consequently, that the whole quotient is between 29 and 30. If we multiply the unit figure of the quotient, 9, by the divisor, 8, and subtract the product from the last partial dividend, 79, the remainder, 7, will evidently be the excess of the dividend, 239, above the product of the factors, 29 and 8. Indeed, having, by the different parts of the operation, subtracted successively from the dividend, 239, the product of each figure of the quotient by the divisor, we have evidently subtracted the product of the whole quotient by the divisor, or 232; and the remainder, 7, less than the divisor, proves, that 232 is the greatest multiple of 8, that can be contained in 239.

50. It must be perceived, after what has been said, that to reproduce any dividend, we must add to the product of the divisor by the quotient, the sum which remains when the divisor cannot be performed exactly.

51. If we wished to divide into eight equal parts a sum of whatever nature, consisting of 239 units, we could not do it without using parts of units or *fractions*. Thus, when we have taken from the number 239, the 8 times 29 units contained in it, there will remain 7 units, to be divided into 8 parts; to do this, we may divide each of these units, one after the other, into 8 parts, and then take one part out of each unit, which will give 7 parts to be joined to the 29 whole units, to form the eighth part of 239, or the exact quotient of this number, by 8.

The same reasoning may be applied to every other example of division in which there is a remainder, and in this case the quotient is composed of two parts; one, consisting of whole units, while the other cannot be obtained, until the concrete or material units of the remainder have been actually divided into the number of parts denoted by the divisor; without this it can only be indicated by supposing, *a unit of the dividend to be divided into as many parts as there are units in the divisor, and so many of these parts, as there are units in the remainder, taken to complete the quotient required.*

52. In general, when we have occasion to consider quantities less than unity, we suppose unity divided into a certain number of parts, sufficiently small to be contained a certain number of times in these quantities, or to *measure* them. In the idea thus formed of their magnitude there are two elements, namely, the number of times the measuring part is contained in unity, and the number of these parts found in the quantities.

A nomenclature has been made for fractions, which answers to this manner of conceiving and representing them.

That which results from the division of unity

into 2 parts is called	a *moiety* or *half*,
into 3 parts	*a third*,
into 4 parts	*a quarter* or *fourth*,
into 5 parts	*a fifth*,
into 6 parts	*a sixth*,

and so on, adding after the two first, the termination *th* to the number, which denotes how many parts are supposed to be in unity.

Every fraction then is expressed by two numbers; the first, which shows how many parts it is composed of, is called the *numerator*, and the other which shows how many of these parts are necessary to form an unit, is called the *denominator*, because the denomination of the fraction is deduced from it. *Five sixths* of an unit is a fraction, the numerator of which is *five*, and the denominator *six*.

The *numerator* and the *denominator* together are called the *two terms* of the fraction.

Figures are used to shorten the expression of fractions, the denominator being written under the numerator, and separated from it by a line,

one third is written $\frac{1}{3}$,
five sixths $\frac{5}{6}$.

53. According to the meaning attached to the words, *numerator* and *denominator*, it is plain, *that a fraction is increased, by increasing its numerator, without changing its denominator;* for this last, as it shows into how many parts unity is divided, determines the magnitude of these parts, which continues the same, while the denominator remains unchanged; and by augmenting the numerator the number of these parts is augmented, and consequently the fraction increased. It is thus, for instance, that $\frac{8}{9}$ exceeds $\frac{7}{9}$, and that $\frac{13}{30}$ exceeds $\frac{11}{30}$.

It follows evidently from this, *that by repeating the numerator 2, 3, or any number of times, without altering the denominator, we repeat, a like number of times, the quantity expressed by the fraction, or in other words multiply it by this number;* for we make 2, 3, or any number of times, as many parts, as it had before, and these parts have remained each of the same value. The fraction $\frac{3}{5}$, then, is the triple of $\frac{1}{5}$, and $\frac{10}{21}$ the double of $\frac{5}{21}$.

A fraction is diminished by diminishing its numerator, without changing its denominator, since it is made to consist of a less number of parts than it contained before, and these parts retain the same value. Whence, *if the numerator be divided by 2, 3, or any number, without the denominator being altered, the fraction is made a like nuuber of times smaller, or is divided by that number,* for it is made to contain 2, 3, or any number of times less parts than it contained before, and these parts remain of the same value. Thus $\frac{1}{5}$ is a third of $\frac{3}{5}$, and $\frac{5}{21}$ is half of $\frac{10}{21}$.

54. On the contrary, a fraction is diminished, when its denominator is increased without changing its numerator; for then more parts are supposed in an unit, and consequently they must be smaller, but, as only the same number of them are taken to form the fraction, the amount in this case must be a less quantity than in the first. Thus $\frac{2}{5}$ is less than $\frac{2}{3}$, and $\frac{4}{13}$ than $\frac{4}{9}$.

Hence it follows, that *if the denominator of a fraction be multiplied by 2, 3, or any number, without the numerator being changed,* *the fraction becomes a like number of times smaller, or is divided by that number,* for it is composed of the same number of parts as before, but each of them has become 2, 3, or a certain number of times less. The fraction $\frac{3}{8}$ is half of $\frac{3}{4}$, and $\frac{4}{15}$ the third of $\frac{4}{5}$.

A fraction is increased when its denominator is diminished without the numerator being changed; because, as unity is supposed to be divided into fewer parts, each one becomes greater, and their amount is therefore greater.

Whence, *if the denominator of a fraction be divided by 2, 3, or any other number, the fraction will be made a like number of times greater, or will be multiplied by that number;* for the number of parts remains the same, and each one becomes 2, 3, or a certain number of times greater than it was before. According to this $\frac{3}{5}$ is triple of $\frac{3}{15}$ and $\frac{5}{6}$ the quadruple of $\frac{5}{24}$.

It may be remarked, that to suppress the denominator of a fraction is the same as to multiply the fraction by that number. For instance, to supress the denominator 3 in the fraction $\frac{2}{3}$ is to change it into 2 whole ones, or to multiply it by 3.

55. The preceding propositions may be recapitulated as follows;

By multiplying By dividing	the numerator, the fraction is	multiplied. divided.
By multiplying By dividing	the denominator, the fraction is	divided. multiplied.

56. The first consequence to be drawn from this table is, that the operations performed on the denominator produce effects of an *inverse* or *contrary* nature with respect to the value of the fraction. Hence it results, *that, if both the numerator and denominator of a fraction be multiplied at the same time, by the same number, the value of the fraction will not be altered;* for if, on the one hand, multiplying the numerator makes the fraction 2, 3, &c. times greater, so on the other, by the second operation, the half or third part, &c. of it is taken; in other words it is divided by the same number, by which it had at first been multiplied. Thus $\frac{1}{5}$ is equal to $\frac{3}{15}$, and $\frac{5}{21}$ is equal to $\frac{10}{42}$.

57. It is also manifest that, *if both the numerator and denominator of a fraction be divided, at the same time, by the same number, the value of the fraction will not be altered;* for if, on the one hand, by dividing the numerator the fraction is made 2, 3, &c.

times smaller; on the other, by the second operation, the double, triple, &c. is taken; in short it is multiplied by the same number, by which it was at first divided. Thus the fraction $\frac{2}{4}$ is equal to $\frac{1}{2}$, and $\frac{3}{9}$ is equal to $\frac{1}{3}$.

58. It is not with fractions as with whole numbers, in which a magnitude, so long as it is considered with relation to the same unit, is susceptible of but one expression. In fractions on the contrary, the same magnitude can be expressed in an infinite number of ways. For instance, the fractions,

$$\frac{1}{2},\ \frac{2}{4},\ \frac{3}{6},\ \frac{4}{8},\ \frac{5}{10},\ \frac{6}{12},\ \frac{7}{14},\ \text{\&c.}$$

in each of which the denominator is twice as great as the numerator, express, under different forms, the half of an unit. The fractions, $\frac{1}{3},\ \frac{2}{6},\ \frac{3}{9},\ \frac{4}{12},\ \frac{5}{15},\ \frac{6}{18},\ \frac{7}{21}$, &c.
of which the denominator is three times as great as the numerator, represent each the third part of an unit. Among all the forms, which the given fraction assumes, in each instance, the first is the most remarkable, as being the most simple; and, consequently, it is well to know how to find it from any of the others. It is obtained by dividing the two terms of the others by the same number, which, as has already been shown, does not alter their value. Thus if we divide by 7 the two terms of the fraction $\frac{7}{14}$, we come back to $\frac{1}{2}$; and, performing the same operation on $\frac{7}{21}$, we get $\frac{1}{3}$.

59. It is by following this process, that a fraction is reduced to its most *simple terms;* it cannot, however, be applied except to fractions, of which the numerator and denominator are divisible by the same number; in all other cases the given fraction is the most simple of all those, that can represent the quantity it expresses. Thus the fractions $\frac{5}{7}$, $\frac{7}{12}$, $\frac{15}{16}$, the terms of which cannot be divided by the same number, or *have no common divisor*, are *irreducible*, and, consequently, cannot express, in a more simple manner, the magnitudes which they represent.

60. Hence it follows, that to simplify a fraction, we must endeavour to divide its two terms by some one of the numbers, 2, 3, &c; but by this uncertain mode of proceeding it will not be always possible to come at the most *simple terms* of the given fraction, or at least, it will often be necessary to perform a great number of operations.

If, for instance, the fraction $\frac{24}{84}$ were given, it may be seen at once, that each of its terms is a multiple of 2, and dividing them by this number, we obtain $\frac{12}{42}$; dividing these last also by 2, we obtain $\frac{6}{21}$. Although much more simple now than at first, this fraction is still susceptible of reduction, for its two terms can be divided by 3, and it then becomes $\frac{2}{7}$.

If we observe, that to divide a number by 2, then the quotient by 2, and then the second quotient by 3, is the same thing as to divide the original number by the product of the numbers, 2, 2, and 3, which amounts to 12, we shall see that the three above operations can be performed at once by dividing the two terms of the given fraction by 12, and we shall again have $\frac{2}{7}$.

The numbers 2, 3, 4, and 12, each dividing the two numbers 24 and 84 at the same time, are the common divisors of these numbers; but 12 is the most worthy of attention, because it is the greatest, and it is by employing *the greatest common divisor* of the two terms of the given fraction, that it is reduced at once to its most simple terms. We have then this important problem to solve, *two numbers being given, to find their greatest common divisor*†.

61. We arrive at the knowledge of the common divisor of two numbers by a sort of trial easily made, and which has this recommendation, that each step brings us nearer and nearer to the number sought. To explain it clearly, I will take an example.

Let the two numbers be 637 and 143. It is plain, that the greatest common divisor of these two numbers cannot exceed the smallest of them; it is proper then to try if the number 143, which divides itself and gives 1 for the quotient, will also divide the number 637, in which case it will be the greatest common divisor sought. In the given example this is not the case; we obtain a quotient 4, and a remainder 65.

Now it is plain, that every common divisor of the two numbers, 143 and 637, ought also to divide 65, the remainder resulting from their division; for the greater, 637, is equal to the

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† What is here called the *greatest common divisor*, is sometimes called the *greatest common measure.*

less, 143, multiplied by 4, plus the remainder, 65, (50); now in dividing 637 by the common divisor sought, we shall have an exact quotient; it follows then, that we must obtain a like quotient, by dividing the assemblage of parts, of which 637 is composed, by the same divisor; but the product of 143 by 4 must necessarily be divisible by the common divisor, which is a factor of 143, and consequently the other part, 65, must also be divisible by the same divisor; otherwise the quotient would be a whole number accompanied by a fraction, and consequently could not be equal to the whole number, resulting from the division of 637 by the common divisor. By the same reasoning, it may be proved in general, *that every common divisor of two numbers must also divide the remainder resulting from the division of the greater of the two by the less.*

According to this principle, we see, that the common divisor of the numbers, 637 and 143, must also be the common divisor of the numbers 143 and 65; but as the last cannot be divided by a number greater than itself, it is necessary to try 65 first. Dividing 143 by 65, we find a quotient 2, and a remainder 13; 65 then is not the divisor sought. By a course of reasoning, similar to that pursued with regard to the numbers, 637, 143, and the remainder, resulting from their division, 65, it will be seen that every common divisor of 143 and 65 must also divide the numbers 65 and 13; now the greatest common divisor of these two last cannot exceed 13, we must therefore try, if 13 will divide 65, which is the case, and the quotient is 5; then 13 is the greatest common divisor sought.

We can make ourselves certain of its possessing this property by resuming the operations in an inverse order, as follows;

As 13 divides 65 and 13, it will divide 143, which consists of twice 65 added to 13; as it divides 65 and 143, it will divide 637, which consists of 4 times 143 added to 65; 13 then is the common divisor of the two given numbers. It is also evident, by the very mode of finding it, that there can be no common divisor greater than 13, since 13 must be divided by it.

It is convenient in practice, to place the successive divisions one after the other, and to dispose of the operation as may be seen in the following example;

637	143	65	13
572	4 | 130	2 | 65	5
65	13	0	

the quotients, 4, 2, 5, being separated from the other figures.

The reasoning employed in the preceding example, may be applied to any numbers, and thus conduct us to this general rule. *The greatest common divisor of two numbers will be found, by dividing the greater by the less; then the less by the remainder of the first division; then this remainder, by the remainder of the second division; then this second remainder by the third, or that of the third division; and so on, till we arrive at an exact quotient; the last divisor will be the common divisor sought.*

62. See two examples of the operation.

9024	3760	1504	752
7520	2 | 3008	2 | 1504	2
1504	752	00	

752 then is the greatest common divisor of 9024 and 3760.

937	47	44	3	2	1
47	19 | 44	1 | 3	14 | 2	1 | 2	2
467	3	14	1	0	
423		12			
44		2			

By this last operation we see that the greatest common divisor of 937 and 47, is 1 only, that is, these two numbers properly speaking have no common divisor, since all whole numbers, like them, are divisible by 1.

We may easily satisfy ourselves, that the rule of the preceding article must necessarily lead to this result, whenever the given numbers have no common divisor; for the remainders, each being less than the corresponding divisor, become less and less every operation, and it is plain, that the division will continue as long as there is a divisor greater than unity.

63. After these calculations, the fractions $\frac{143}{637}$ and $\frac{3760}{9024}$, can be at once reduced to their most simple terms, by dividing the terms of the first by their common divisor, 13, and the terms of the second, by their common divisor, 752; we thus obtain $\frac{11}{49}$

and $\frac{5}{12}$. As to the fraction, $\frac{47}{937}$, it is altogether irreducible, since its terms have no common divisor but unity.

64. It is not always necessary to find the greatest common divisor of the given fraction; there are, as has before been remarked, reductions, which present themselves without this preparatory step.

Every number terminated by one of the figures 0, 2, 4, 6, 8, is necessarily divisible by 2; for in dividing any number by 2, only 1 can remain from the tens; the last partial division can be performed on the numbers 0, 2, 4, 6, 8, if the tens leave no remainder, and on the numbers 10, 12, 14, 16, 18, if they do, and all these numbers are divisible by 2.

The numbers divisible by 2, are called *even numbers*, because they can be divided into two equal parts.

Also, every number terminated on the right by a cipher, or by 5, is divisible by 5, for when the division of the tens by 5 has been performed, the remainder, if there be one, must necessarily be either 1, 2, 3, or 4, the remaining part of the operation will be performed on the numbers 0, 5, 10, 15, 20, 25, 30, 35, 40, or 45, all of which are divisible by 5.

The numbers, 10, 100, 1000, &c. expressed by unity followed by a number of ciphers, can be resolved into 9 added to 1, 99 added to 1, 999 added to 1, and so on; and the numbers, 9, 99, 999, &c. being divisible by 3, and by 9, it follows that, if numbers of the form 10, 100, 1000, &c. be divided by 3 or 9, the remainder of the division will be 1.

Now every number, which, like 20, 300 or 5000, is expressed by a single significant figure followed on the right, by a number of ciphers, can be resolved into several numbers expressed by unity, followed on the right by a number of ciphers; 20 is equal to 10 added to 10; 300, to 100 added to 100 added to 100; 5000, to 1000 added to 1000 added to 1000 added to 1000 added to 1000; and so with others. Hence it follows, that if 20, or 10 added to 10, be divided by 3 or 9, the remainder will be 1 added to 1, or 2; if 300, or 100 added to 100 added to 100, be divided by 3 or 9, the remainder will be 1 added to 1 added to 1, or 3.

In general, if we resolve in the same manner a number expressed by one significant figure, followed, on the right, by a number of ciphers, in order to divide it by 3 or 9; the remainder of this division will be equal to as many times 1, as there are units in the significant figure, that is, it will be equal to the significant figure itself. Now any number being resolved into units, tens, hundreds, &c. is formed by the union of several numbers expressed by a single significant figure; and, if each of these last be divided by 3 or 9, the remainder will be equal to one of the significant figures of the given number; for instance, the division of hundreds will give, for a remainder, the figure occupying the place of hundreds; that of tens, the figure occupying the place of tens; and so of the others. If then, the sum of all these remainders be divisible by 3 or 9, the division of the given number by 3 or 9 can be performed exactly; whence it follows, that if the sum of the figures, constituting any number, be divisible by 3 or 9, the number itself is divisible by 3 or 9.

Thus the numbers, 423, 4251, 15342, are divisible by 3, because the sum of the significant figures is 9 in the first, 12 in the second, and 15 in the third.

Also, 621, 8280, 934218, are divisible by 9, because the sum of the significant figures is 9 in the first, 18 in the second, and 27 in the third.

It must be observed, that every number divisible by 9 is also divisible by 3, although every number divisible by 3, is not also divisible by 9.

Observations might be made on several other numbers analogous to those just given on 2, 3, 5 and 9; but this would lead me too far from the subject.

The numbers 1, 3, 5, 7, 11, 13, 17, &c. which can be divided only by themselves, and by unity, are called *prime numbers;* two numbers, as 12 and 35, having, each of them, divisors, but neither of them any one, that is common to it with the other, are called *prime to each other.*

Consequently, the numerator and denominator of an irreducible fraction are prime to each other.

Examples for practice under Article 61.

What is the greatest common divisor of 24 and 36? *Ans.* 12.

What is the greatest common divisor of 35 and 100? *Ans.* 5.

What is the greatest common divisor of 312 and 504?

Ans. 24.

Examples for practice under articles 57, 58 and 60.

Reduce $\frac{25}{75}$ to its most simple terms. *Ans.* $\frac{1}{3}$.

Reduce $\frac{512}{4096}$ to its most simple terms. *Ans.* $\frac{1}{8}$.

Reduce $\frac{81}{729}$ to its most simple terms. *Ans.* $\frac{1}{9}$.

Reduce $\frac{160}{168}$ to its most simple terms. *Ans.* $\frac{20}{21}$.

Reduce $\frac{324}{378}$ to its most simple terms. *Ans.* $\frac{6}{7}$.

Reduce $\frac{2640}{2880}$ to its most simple terms. *Ans.* $\frac{11}{12}$.

65. After this digression we will resume the examination of the table in article 55.

$$\left.\begin{array}{l}\text{By multiplying}\\ \text{By dividing}\end{array}\right\}\text{ the numerator, the fraction is }\left\{\begin{array}{l}\text{multiplied,}\\ \text{divided,}\end{array}\right.$$

$$\left.\begin{array}{l}\text{By multiplying}\\ \text{By dividing}\end{array}\right\}\text{ the denominator, the fraction is }\left\{\begin{array}{l}\text{divided,}\\ \text{multiplied,}\end{array}\right.$$

that we may deduce from it some new inferences.

We see at once, by an inspection of this table, that a fraction can be multiplied in two ways, namely, by multiplying its numerator, or dividing its denominator, and that, it can also be divided in two ways, namely, by dividing its numerator, or multiplying its denominator; hence it follows, that multiplication alone, according as it is performed on the numerator or denominator, is sufficient for the multiplication and division of fractions by whole numbers. Thus $\frac{3}{15}$, multiplied by 7 units, makes $\frac{21}{15}$; $\frac{4}{9}$, divided by 3, makes $\frac{4}{27}$.

Examples for practice.

Multiply $\frac{2}{3}$ by 5.	*Ans.* $\frac{10}{3}$.	Divide $\frac{3}{6}$ by 3.	*Ans.* $\frac{1}{6}$.
Multiply $\frac{4}{21}$ by 4.	*Ans.* $\frac{16}{21}$.	Divide $\frac{4}{18}$ by 6.	*Ans.* $\frac{1}{27}$.
Multiply $\frac{3}{48}$ by 6.	*Ans.* $\frac{3}{8}$.	Divide $\frac{5}{6}$ by 10.	*Ans.* $\frac{1}{12}$.
Multiply $\frac{5}{9}$ by 30.	*Ans.* $\frac{150}{9}$.	Divide $\frac{7}{9}$ by 8.	*Ans.* $\frac{7}{72}$.
Multiply $\frac{1}{30}$ by 5.	*Ans.* $\frac{1}{6}$.	Divide $\frac{20}{25}$ by 4.	*Ans.* $\frac{1}{5}$.
Multiply $\frac{2}{45}$ by 9.	*Ans.* $\frac{2}{5}$.	Divide $\frac{22}{11}$ by 4.	*Ans.* $\frac{1}{2}$.

66. The doctrine of fractions enables us to generalize the difinition of multiplication given in article 21. When the multi-

plier is a whole number, it shows how many times the multiplicand is to be repeated; but the term multiplication, extended to fractional expressions, does not always imply augmentation, as in the case of whole numbers. To comprehend in one statement every possible case, it may be said, *that to multiply one number by another is, to form a number by means of the first, in the same manner as the second is formed, by means of unity.* In reality, when it is required to multiply by 2, by 3, &c. the product consists of twice, three times, &c. the multiplicand, in the same way as the multiplier consists of two, three, &c. units; and to multiply any number by a fraction $\frac{1}{5}$ for example, is to take the fifth part of it, because the multiplier $\frac{1}{5}$, being the fifth part of unity, shows that the product ought to be the fifth part of the multiplicand*.

Also, to multiply any number by $\frac{4}{5}$ is to take out of this number or the multiplicand, a part, which shall be four fifths of it, or equal to four times one fifth.

Hence it follows, *that the object in multiplying by a fraction, whatever may be the multiplicand, is, to take out of the multiplicand a part, denoted by the multiplying fraction;* and that this operation is composed of two others, namely, a division and a multiplication, in which the divisor and multiplier are whole numbers.

Thus, for instance, to take $\frac{4}{5}$ of any number, it is first necessary to find the fifth part, by dividing the number by 5, and to repeat this fifth part four times, by multiplying it by 4.

We see, in general, *that the multiplicand must be divided by the denominator of the multiplying fraction, and the quotient be multiplied by its numerator.*

The multiplier being less than unity, the product will be smaller than the multiplicand, to which it would be only equal, if the multiplier were 1.

67. If the multiplicand be a whole number divisible by 5, for

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* We are led to this statement, by a question which often presents itself; namely, where the price of any quantity of a thing is required, the price of the unity of the thing being known. The question evidently remains the same, whether the given quantity be greater or less than this unity.

instance, 35, the fifth part will be 7; this result, multiplied by 4, will give 28 for the $\frac{4}{5}$ of 35, or for the product of 35 by $\frac{4}{5}$. If the multiplicand, always a whole number, be not exactly divisible by 5, as, for instance, if it were 32, the division by 5 will give for a quotient $6\frac{2}{5}$; this quotient repeated 4 times will give $24\frac{8}{5}$.

This result presents a fraction in which the numerator exceeds the denominator, but this may be easily explained. The expression $\frac{8}{5}$, in reality denoting 8 parts, of which 5, taken together, make unity, it follows, that $\frac{8}{5}$ is equivalent to unity added to three fifths of unity, or $1\frac{3}{5}$; adding this part to the 24 units, we have $25\frac{3}{5}$ for the value of $\frac{4}{5}$ of 32.

68. It is evident, from the preceding example, that the fraction $\frac{8}{5}$ contains unity, or *a whole one,* and $\frac{3}{5}$, and the reasoning, which led to this conclusion, shows also, that every fractional expression, of which the numerator exceeds the denominator, contains one or more units, or whole ones, and *that these whole ones may be extracted by dividing the numerator by the denominator; the quotient is the number of units contained in the fraction, and the remainder, written as a fraction, is that, which must accompany the whole ones.*

The expression $\frac{307}{53}$, for instance, denoting 307 parts, of which 53 make unity, there are, in the quantity represented by this expression, as many whole ones, as the number of times 53 is contained in 307; if the division be performed, we shall obtain 5 for the quotient, and 42 for the remainder, these 42 are fifty third parts of unity; thus, instead of $\frac{307}{53}$, may be written $5\frac{42}{53}$.

Examples for practice.

Reduce the fraction $\frac{6}{3}$ to its equivalent whole number.

Ans. 2.

Reduce $\frac{7}{2}$ to its equivalent whole or mixed number. *Ans.* $3\frac{1}{2}$.

Reduce $\frac{15}{4}$ to its equivalent whole or mixed number.

Ans. $3\frac{3}{4}$.

Reduce $\frac{482}{20}$ to its equivalent whole or mixed number.

Ans. $24\frac{2}{20}$.

Reduce $\frac{97}{8}$ to its equivalent whole or mixed number.

Ans. $12\frac{1}{8}$.

Reduce $\frac{512}{50}$ to its equivalent whole or mixed number.

Ans. $10\frac{6}{25}$.

69. The expression $5\frac{42}{53}$, in which the whole number is given, being composed of two different parts, we have often occasion to convert it into the original expression $\frac{307}{53}$, which is called, *reducing a whole number to a fraction.*

To do this, the *whole number is to be multiplied by the denominator of the accompanying fraction, the numerator to be added to the product, and the denominator of the same fraction to be given to the sum.*

In this case, the 5 whole ones must be converted into fifty-thirds, which is done by multiplying 53 by 5, because each unit must contain 53 parts; the result will be $\frac{265}{53}$; joining this part with the second, $\frac{42}{53}$, the answer will be $\frac{307}{53}$.

Examples for practice.

Reduce $12\frac{1}{2}$ to a fraction. *Ans.* $\frac{25}{2}$.

Reduce $6\frac{5}{9}$ to a fraction. *Ans.* $\frac{59}{9}$.

Reduce $31\frac{7}{10}$ to a fraction. *Ans.* $\frac{317}{10}$.

Reduce $45\frac{21}{130}$ to a fraction. *Ans.* $\frac{5871}{130}$.

70. We now proceed to the multiplication of one fraction by another.

If, for instance, $\frac{2}{3}$ were to be multiplied by $\frac{4}{5}$; according to article 66, the operation would consist in dividing $\frac{2}{3}$ by 5, and multiplying the result by 4; according to the table in article 65, the first operation is performed by multiplying 3, the denominator of the multiplicand, by 5; and the second, by multiplying 2, the numerator of the multiplicand, by 4; and the required product is thus found to be $\frac{8}{15}$.

It will be the same with every other example, and it must consequently be concluded from what precedes, *that to obtain the product of two fractions, the two numerators must be multiplied, one by the other, and under the product must be placed the product of the denominators.*

Examples.

Multiply $\frac{1}{5}$ by $\frac{3}{4}$. *Ans.* $\frac{3}{20}$. Multiply $\frac{4}{9}$ by $\frac{2}{7}$. *Ans.* $\frac{8}{63}$.

Multiply $\frac{5}{6}$ by $\frac{3}{8}$. *Ans.* $\frac{5}{16}$. Multiply $\frac{25}{30}$ by $\frac{10}{12}$. *Ans.* $\frac{25}{36}$.
Multiply $\frac{7}{21}$ by $\frac{1}{2}$. *Ans.* $\frac{1}{6}$. Multiply $\frac{11}{13}$ by $\frac{31}{42}$. *Ans.* $\frac{341}{546}$.

71. It may sometimes happen that two mixed numbers, or whole numbers joined with fractions, are to be multiplied, one by the other, as for instance, $3\frac{5}{7}$ by $4\frac{8}{9}$. The most simple mode of obtaining the product is, to reduce the whole numbers to fractions by the process in article 69; the two factors will then be expressed by $\frac{26}{7}$ and $\frac{44}{9}$, and their product, by $\frac{1144}{63}$ or $18\frac{10}{63}$, by extracting the whole ones (68).

72. The name *fractions of fractions* is sometimes given to the product of several fractions; in this sense we say, $\frac{2}{3}$ of $\frac{4}{5}$ This expression denotes $\frac{2}{3}$ of the quantity represented by $\frac{4}{5}$ of the original unit, and taken in its stead for unity. These two fractions are reduced to one by multiplication (70), and the result, $\frac{8}{15}$, expresses the value of the quantity required, with relation to the original unit; that is, $\frac{2}{3}$ of the quantity represented by $\frac{4}{5}$ of unity is equivalent to $\frac{8}{15}$ of unity. If it were required to take $\frac{7}{9}$ of this result, it would amount to taking $\frac{7}{9}$ of $\frac{2}{3}$ of $\frac{4}{5}$. and these fractions, reduced to one, would give $\frac{56}{135}$ for the value of the quantity sought, with relation to the original unit.

73. The word *contain,* in its strict sense, is not more proper in the different cases presented by division, than the word *repeat* in those presented by multiplication; for it cannot be said that the dividend contains the divisor, when it is less than the latter; the expression is generally used, but only by analogy and extension.

To generalize division, *the dividend must be considered as having the same relation to the quotient, that the divisor has to unity,* because the divisor and quotient are the two factors of the dividend (36). This consideration is conformable to every case that division can present. When, for instance, the divisor is 5, the dividend is equal to 5 times the quotient, and, consequently, this last is the fifth part of the dividend. If the divisor be a fraction, $\frac{1}{2}$ for instance, the dividend cannot be but half of the quotient, or the latter must be double the former.

The definition, just given, easily suggests the mode of proceeding, when the divisor is a fraction. Let us take, for example, $\frac{4}{5}$. In this case the dividend ought to be only $\frac{4}{5}$ of the quotient; but $\frac{1}{5}$ being $\frac{1}{4}$ of $\frac{4}{5}$, we shall have $\frac{1}{5}$ of the quotient, by taking $\frac{1}{4}$ of the dividend, or dividing it by 4. Thus knowing $\frac{1}{5}$ of the quotient, we have only to take it 5 times, or multiply it by 5, to obtain the quotient. In this operation the dividend is divided by 4 and multiplied by 5, which is the same as taking $\frac{5}{4}$ of the dividend, or multiplying it by $\frac{5}{4}$, which fraction is no other than the divisor inverted.

This example shows, that, in general, *to divide any number by a fraction, it must be multiplied by the fraction inverted.*

For instance, let it be required to divide 9 by $\frac{3}{4}$; this will be done by multiplying it by $\frac{4}{3}$. and the quotient will be found to be $\frac{36}{3}$ or 12. Also 13 divided by $\frac{5}{7}$ will be the same as 13 multiplied by $\frac{7}{5}$, or $\frac{91}{5}$. The required quotient will be $18\frac{1}{5}$, by extracting the whole ones (68).

It is evident that, whenever the numerator of the divisor is less than the denominator, the quotient will exceed the dividend, because the divisor in that case, being less than unity, must be contained in the dividend a greater number of times, than unity is, which, taken for a divisor, always gives a quotient exactly the same as the dividend.

74. *When the dividend is a fraction, the operation must be performed by multiplying the dividend by the divisor inverted* (70).

Let it be required to divide $\frac{7}{8}$ by $\frac{2}{3}$; according to the preceding article, $\frac{7}{8}$ must be multiplied by $\frac{3}{2}$, which gives $\frac{21}{16}$.

It is evident, that the above operation may be enunciated thus; *To divide one fraction by another, the numerator of the first must be multiplied by the denominator of the second, and the denominator of the first, by the numerator of the second.*

If there be whole numbers joined to the given fractions, they must be reduced to fractions, and the above rule applied to the results.

Examples.

Divide 9 by $\frac{2}{5}$.	*Ans.* $\frac{45}{2}$.	Divide $7\frac{1}{2}$ by $\frac{1}{3}$.	*Ans.* $\frac{45}{2}$.
Divide 18 by $\frac{6}{5}$.	*Ans.* 15.	Divide $2\frac{2}{3}$ by $3\frac{1}{4}$.	*Ans.* $\frac{32}{39}$.
Divide $\frac{3}{6}$ by $\frac{7}{9}$.	*Ans.* $\frac{9}{14}$.	Divide $\frac{63}{9}$ by $\frac{9}{63}$.	*Ans.* 42.
Divide $\frac{10}{11}$ by $\frac{4}{30}$.	*Ans.* $\frac{75}{11}$.	Divide $\frac{44}{11}$ by $\frac{44}{11}$.	*Ans.* 1.

75. It is important to observe, that any division, whether it can be performed in whole numbers or not, may be indicated by a fractional expression; $\frac{36}{3}$, for instance, expresses evidently the quotient of 36 by 3, as well as 12, for $\frac{1}{3}$ being contained three times in unity, $\frac{36}{3}$ will be contained 3 times in 36 units, as the quotient of 36 by 3 must be.

76. It may seem preposterous to treat of the multiplication and division of fractions before having said any thing of the manner of adding and subtracting them; but this order has been followed, because multiplication and division follow as the immediate consequences of the remark given in the table of article 55, but addition and subtraction require some previous preparation. It is, besides, by no means surprising, that it should be more easy to multiply and divide fractions, than to add and subtract them, since they are derived from division, which is so nearly related to multiplication. There will be many opportunities, in what follows, of becoming convinced of this truth; that operations to be performed on quantities are so much the more easy, as they approach nearer to the origin of these quantities. We will now proceed to the addition and subtraction of fractions.

77. When the fractions on which these operations are to be performed have the same denominator, as they contain none but parts of the same denomination, and consequently of the same magnitude or value, they can be added or subtracted in the same manner as whole numbers, care being taken to mark, in the result, the denomination of the parts, of which it is composed.

It is indeed very plain, that $\frac{2}{11}$ and $\frac{3}{11}$ make $\frac{5}{11}$, as 2 quantities and 3 quantities, of the same kind, make 5 of that kind, whatever it may be.

Also, the difference between $\frac{3}{9}$ and $\frac{8}{9}$ is $\frac{5}{9}$, as the difference between 3 quantities and 8 quantities, of the same kind, is 5 of that kind, whatever it may be. Hence it must be concluded, that, *to add or subtract fractions, having the same denominator, the sum or difference of their numerators must be taken, and the common denominator written under the result.*

78. When the given fractions have different denominators, it

is impossible to add together, or subtract, one from the other, the parts of which they are composed, because these parts are of different magnitudes; but to obviate this difficulty, the fractions are made to undergo a change, which brings them to parts of the same magnitude, by giving them a common denominator.

For instance, let the fractions be $\frac{2}{3}$ and $\frac{4}{5}$; if each term of the first be multiplied by 5, the denominator of the second, the first will be changed into $\frac{10}{15}$; and if each term of the second be multiplied by 3, the denominator of the first, the second will be changed into $\frac{12}{15}$; thus too new expressions will be formed, having the same value as the given fractions (56).

This operation, necessary for comparing the respective magnitudes of two fractions, consists simply in finding, to express them, parts of an unit sufficiently small to be contained exactly in each of those which form the given fractions. It is plain, in the above example, that the fifteenth part of an unit will exactly measure $\frac{1}{3}$, and $\frac{1}{5}$ of this unit, because $\frac{1}{3}$ contains five 15ths, and $\frac{1}{5}$ contains three 15ths. The process, applied to the fractions $\frac{2}{3}$ and $\frac{4}{5}$, will admit of being applied to any others.

In general, *to reduce any two fractions to the same denominator, the two terms of each of them must be multiplied by the denominator of the other.*

79. *Any number of fractions are reduced to a common denominator, by multiplying the two terms of each by the product of the denominators of all the others;* for it is plain that the new denominators are all the same, since each one is the product of all the original denominators, and that the new fractions have the same value as the former ones, since nothing has been done except multiplying each term of these by the same number (56).

Examples.

Reduce $\frac{3}{4}$ and $\frac{5}{9}$ to a common denominator. *Ans.* $\frac{27}{36}$, $\frac{20}{36}$.

Reduce $\frac{8}{10}$ and $\frac{3}{7}$ to a common denominator. *Ans.* $\frac{56}{70}$, $\frac{30}{70}$.

Reduce $\frac{1}{3}$, $\frac{3}{4}$ and $\frac{4}{5}$ to a common denominator. *Ans.* $\frac{20}{60}$, $\frac{45}{60}$, $\frac{48}{60}$.

Reduce $\frac{2}{10}$, $\frac{3}{5}$, $\frac{4}{7}$ and $\frac{5}{9}$ to a common denominator.

Ans. $\frac{630}{3150}$, $\frac{1890}{3150}$, $\frac{1800}{3150}$, $\frac{1750}{3150}$.

The preceding rule conducts us, in all cases, to the proposed end; but when the denominators of the fractions in question are not prime to each other, there is a common denominator more simple than that which is thus obtained, and which may be shown to result from considerations analogous to those given in the preceding articles. If, for instance, the fractions were $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, as nothing more is required, for reducing them to a common denominator, than to divide unity into parts, which shall be exactly contained in those of which these fractions consist, it will be sufficient to find the smallest number, which can be exactly divided by each of their denominators, 3, 4, 6, 8; and this will be discovered by trying to divide the multiples of 3 by 4, 6, 8; which does not succeed until we come to 24, when we have only to change the given fractions into 24$^{\text{ths}}$ of an unit.

To perform this operation we must ascertain successively how many times the denominators, 3, 4, 6 and 8, are contained in 24, and the quotients will be the numbers, by which each term of the respective fractions must be multiplied, to be reduced to the common denominator, 24. It will thus be found, that each term of $\frac{2}{3}$ must be multiplied by 8, each term of $\frac{3}{4}$ by 6, each term of $\frac{5}{6}$ by 4, and each term of $\frac{7}{8}$ by 3; the fractions will then become $\frac{16}{24}$, $\frac{18}{24}$, $\frac{20}{24}$, $\frac{21}{24}$.

Algebra will furnish the means of facilitating the application of this process.

80. By reducing fractions to the same denominator, they may be added and subtracted as in article 77.

81. When there are at the same time both whole numbers and fractions, the whole numbers, if they stand alone, must be converted into fractions of the same denomination as those, which are to be added to them, or subtracted from them; and if the whole numbers are accompanied with fractions, they must be reduced to the same denominator with these fractions.

It is thus, that the addition of 4 units and $\frac{2}{3}$ changes itself into the addition of $\frac{12}{3}$ and $\frac{2}{3}$, and gives for the result $\frac{14}{3}$.

To add $3\frac{2}{7}$ to $5\frac{4}{9}$, the whole numbers must be reduced to fractions, of the same denomination as those which accompany them, which reduction gives $\frac{23}{7}$ and $\frac{49}{9}$; with these results the sum is found to be $\frac{550}{63}$, or $8\frac{46}{63}$. If, lastly, $\frac{4}{5}$ were to be subtracted from $3\frac{1}{4}$, the operation would be reduced to taking $\frac{4}{5}$ from $\frac{13}{4}$, and the remainder would be $\frac{49}{20}$.

Examples in addition of fractions.

Add $\frac{2}{3}$ to $\frac{3}{9}$. *Ans.* $\frac{27}{27}$, or 1.
Add $\frac{5}{7}$ to $\frac{15}{20}$. *Ans.* $\frac{41}{28}$.
Add $\frac{3}{7}$ to $\frac{5}{2}$. *Ans.* $\frac{41}{14}$.
Add $\frac{3}{7}$, $\frac{4}{2}$ and $\frac{3}{5}$ together. *Ans.* $3\frac{1}{35}$.
Add $2\frac{1}{2}$ to $4\frac{3}{4}$ and $5\frac{1}{3}$ together. *Ans.* $12\frac{7}{12}$.
Add $\frac{4}{5}$, $1\frac{1}{5}$, and $6\frac{3}{5}$, together. *Ans.* $8\frac{3}{5}$.

Examples in subtraction of fractions.

From $\frac{2}{3}$ take $\frac{1}{3}$. *Ans.* $\frac{1}{3}$. From $5\frac{3}{8}$ take $2\frac{1}{2}$. *Ans.* $2\frac{7}{8}$.
From $\frac{3}{4}$ take $\frac{5}{9}$. *Ans.* $\frac{7}{36}$. From $8\frac{2}{3}$ take $4\frac{1}{5}$. *Ans.* $4\frac{7}{15}$.
From $\frac{13}{20}$ take $\frac{4}{10}$. *Ans.* $\frac{1}{4}$. From $3\frac{7}{9}$ take $2\frac{10}{11}$. *Ans.* $\frac{86}{99}$.

82. The rule given, for the reduction of fractions to a common denominator supposes, that a product resulting from the successive multiplication of several numbers into each other, does not vary, in whatever order these multiplications may be performed; this truth, though almost always considered as self-evident, needs to be proved.

We shall begin with showing, that to multiply one number by the product of two others is the same thing as to multiply it at first by one of them, and then to multiply that product by the other. For instance, instead of multiplying 3 by 35, the product of 7 and 5, it will be the same thing, if we multiply 3 by 5, and then that product by 7. The proposition will be evident, if, instead of 3, we take an unit; for 1, multiplied by 5, gives 5, and the product of 5 by 7 is 35, as well as the product of 1 by 35; but 3, or any other number, being only an assemblage of several units, the same property will belong to it, as to each of the units of which it consists; that is, the product of 3 by 5 and by 7, obtained in either way, being the triple of the results given by unity, when multiplied by 5 and 7, must necessarily be the same. It may be proved in the same manner, that were it required to multiply 3 by the product of 5, 7 and 9, it would consist in multiplying 3 by 5, then this product by 7, and the result by 9, and so on, whatever might be the number of factors.

To represent in a shorter manner several successive multiplications, as of the numbers 3, 5, and 7 into each other, we shall write 3 by 5 by 7.

This being laid down, in the product 3 by 5, the order of the factors, 3 and 5 (27), may be changed, and the same product obtained. Hence it directly follows, that 5 by 3 by 7 is the same as 3 by 5 by 7.

The order of the factors 3 and 7, in the product 5 by 3 by 7, may also be changed, because this product is equivalent to 5 multiplied by the product of the numbers 3 and 7; thus we have in the expression 5 by 7 by 3, the same product as the preceding.

By bringing together the three arrangements,

3 by 5 by 7
5 by 3 by 7
5 by 7 by 3,

we see that the factor 3 is found successively, the first, the second, and the third, and that the same may take place with respect to either of the others. From this example, in which the particular value of each number has not been considered, it must be evident, that a product of three factors does not vary, whatever may be the order in which they are multiplied.

If the question were concerning the product of four factors, such as 3 by 5 by 7 by 9, we might, according to what has been said, arrange, as we pleased, the three first or the three last, and thus make any one of the factors pass through all the places. Considering then one of the new arrangements, for instance, this 5 by 7 by 3 by 9, we might invert the order of the two last factors, which would give 5 by 7 by 9 by 3, and would put 3 in the last place. This reasoning may be extended without difficulty to any number of factors whatever.

Decimal Fractions.

83. Although we can, by the preceding rules, apply to fractions, in all cases, the four fundamental operations of arithmetic,

yet it must have been long since perceived, that, if the different subdivisions of a unit. employed for measuring quantities smaller than this unit, had been subjected to a common law of decrease, the calculus of fractions would have been much more convenient, on account of the facility with which we might convert one into another. By making this law of decrease conform to the basis of our system of numeration, we have given to the calculus the greatest degree of simplicity, of which it is capable.

We have seen in article 5, that each of the collections of units, contained in a number, is composed of ten units of the preceding order, as the ten consists of simple units; but there is nothing to prevent our regarding this simple unit, as containing ten parts, of which each one shall be a *tenth*; the tenth as containing ten parts, of which each one shall be a *hundredth* of unity, the hundredth as containing ten parts, of which each one shall be a *thousandth* of unity, and so on.

Proceeding thus, we may form quantities as small as we please, by means of which it will be possible to measure any quantities, however minute. These fractions, which are called *decimals*, because they are composed of parts of unity, that become continually ten times smaller, as they depart further from unity, may be converted, one into the other, in the same manner as *tens, hundreds, thousands*, &c. are converted into units; thus,

the unit being equivalent to	10 tenths,
the tenth	10 hundredths,
the hundredth	10 thousandths,

it follows, that the tenth is equivalent to 10 times 10 thousandths, or 100 thousandths.

For instance, 2 tenths, 3 hundredths and 4 thousandths will be equivalent to 234 thousandths, as 2 hundreds, 3 tens and 4 units make 234 units; and what is here said may be applied universally, since the subordination of the parts of unity is like that of the different orders of units.

84. According to this remark, we can, by means of figures, write decimal fractions in the same manner as whole numbers, since by the nature of our numeration, which makes the value of a figure, placed on the right of another, ten times smaller, *tenths*

naturally take their place on the right of units, then *hundredths* on the right of tenths, and so on; but, that the figures expressing decimal parts may not be confounded with those expressing whole units, a comma† is placed on the right of units. To express, for instance, 34 units and 27 hundredths, we write 34,27. If there be no units, their place is supplied by a cipher, and the same is done for all the decimal parts, which may be wanting between those enunciated in the given number.

Thus 19 hundredths are written 0,19,
304 thousandths 0,304,
3 thousandths 0,003.

85. If the expressions for the above decimal fractions be compared with the following, $\frac{19}{100}$, $\frac{304}{1000}$, $\frac{3}{1000}$, drawn from the general manner of representing a fraction, it will be seen, that *to represent in an entire form a decimal fraction, written as a vulgar fraction, the numerator of the fraction must be taken as it is, and placed after the comma in such a manner, that it may have as many figures as there are ciphers after the unit in the denominator.*

Reciprocally, *to reduce a decimal fraction, given in the form of a whole number, to that of a vulgar fraction, the figures, that it contains, must receive, for a denominator, an unit followed by as many ciphers, as there are figures after the comma.*

Thus the fractions, 0,56, 0,036, are changed into $\frac{56}{100}$ and $\frac{36}{1000}$.

86. *An expression, in figures, of numbers containing decimal parts, is read by enunciating, first, the figures placed on the left of the point, then those on the right, adding to the last figure of the latter the denomination of the parts, which it represents.*

The number 26,736 is read 26 and 736 thousandths;
the number 0,0673 is read 673 ten thousandths,
and 0,0000673 is read 673 ten millionths.

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† In English books on mathematics, and in those that have been written in the United States, decimals are usually denoted by a point, thus 0·19; but the comma is on the whole in the most general use; it is accordingly adopted in this and the subsequent treatises to be published at Cambridge.

87. As decimal figures take their value entirely from their position relative to the comma, it is of no consequence whether we write or omit any number of ciphers on their right. For instance, 0,5 is the same as 0,50; and 0,784 is the same as 0,78400; for, in the first instance, the number, which expresses the decimal fraction, becomes by the addition of a 0 ten times greater, but the parts become hundredths, and consequently on this account are ten times less than before; in the second instance, the number, which expresses the fraction, becomes a hundred times greater than before, but the parts become hundred thousandths, and, consequently are a hundred times smaller than before. This transformation, then, becomes the same as that which takes place with respect to a vulgar fraction, when each of its terms is multiplied by the same number; and if the ciphers be suppressed, it is the same as dividing them by the same number.

88. The addition of decimal fractions and numbers accompanying them, needs no other rule than that given for whole numbers, since the decimal parts are made up one from the other, ascending from right to left, in the same manner as whole units.

For instance, let there be the numbers 0,56, 0,003, 0,958; disposing them as follows,

	0,56
	0,003
	0,958
Sum	1,521

we find, by the rule of article 12, that their sum is 1,521.

Again, let there be the numbers 19,35, 0,3, 48,5, and 110,02, which contain also whole units, they will be disposed thus;

	19,35
	0,3
	48,5
	110,02
Sum	178,17

and their sum will be 178,17.

In general, *the addition of decimal numbers is performed like* *that of whole numbers, care being taken to place the comma in the sum, directly under the commas in the numbers to be added.*

Examples for practice.

Add 4,003, 54,9, 3,21, 6,7203. *Ans.* 68,8333.

Add 409,903, 107,7842, 6,1043, 10,2974. *Ans.* 534,0889.

Add 427, 603,04, 210,15, 3,364, ,021. *Ans.* 1243,575.

89. The rules prescribed for the subtraction of whole numbers, apply also, as will be seen, to decimals. For instance, let 0,3697 be taken from 0,62; it must first be observed, that the second number, which contains only hundredths, while the other contains ten thousandths, can be converted into ten thousandths by placing two ciphers on its right (87), which changes it into 0,6200.

The operation will then be arranged thus;

```
             0,6200
             0,3697
             ------
Difference   0,2503
```

and, according to the rule of article 17, the difference will be 0,2503.

Again, let 7,364 be taken from 9,1457; the operation being disposed thus;

```
             9,1457
             7,3640
             ------
Difference   1,7817
```

the above difference is found. It would have been just as well if no cipher had been placed at the end of the number to be subtracted, provided its different figures had been placed under the corresponding orders of units or parts, in the upper.

In general, *the subtraction of decimal numbers is performed like that of whole numbers, provided that the number of decimal figures, in the two given numbers, be made alike, by writing on the right of that, which has the least, as many ciphers as are necessary; and that the comma in the difference is put directly under those of the given numbers.*

Examples for practice.

From 304,567 take 158,632. *Ans.* 145,935.
From 215,003 take 1,1034. *Ans.* 213,8996.
From 1 take ,9993. *Ans.* 0,0007.
From 68,8333 take ,00042. *Ans.* 68,83288.

The methods of proving addition and subtraction of decimals are the same as those for the addition and subtraction of whole numbers.

90. As the comma separates the collections of entire units from the decimal parts, by altering its place we necessarily change the value of the whole. By moving it towards the right, figures, which were contained in the fractional part, are made to pass into that of whole numbers, and consequently the value of the given number is increased. On the contrary, by moving the comma towards the left, figures, which were contained in the part of whole numbers, are made to pass into that of fractions, and consequently the value of the given number is diminished.

The first change makes the given number, ten, a hundred, a thousand, &c. times greater than before, according as the comma is removed one, two, three, &c. places towards the right, because for each place that the comma is thus removed, all the figures advance with respect to this comma one place towards the left, and consequently assume a value ten times greater than they had before.

If, for example, in the number 134,28 , the point be placed between the 2 and the 8, we shall have 1342,8 , the hundreds will have become thousands, the tens hundreds, the units tens, the tenths units, and the hundredths tenths. Every part of the number having thus become ten times greater, the result is the same as if it had been multiplied by ten.

The second change makes the given number ten, a hundred, a thousand, &c. times smaller than it was before, according as the comma is removed one, two, three, &c. places towards the left, because for each place that the comma is thus removed, all the figures recede, with respect to this comma, one place further to the right, and consequently have a value ten times less than they had before.

If, in the number 134,28, the point be placed between the 3 and 4, we shall have 13,428; the hundreds will become tens, the tens units, the units tenths, the tenths hundredths, and the hundredths thousandths; every part of the number having thus become ten times smaller, the result is the same as if a tenth part of it had been taken, or as if it had been divided by ten.

91. From what has been said, it will be easy to perceive the advantage, which decimal fractions have over vulgar fractions; all the multiplications and divisions, which are performed by the denominator of the latter, are performed with respect to the former, by the addition or suppression of a number of ciphers, or by simply changing the place of the comma. By adapting these modifications to the theory of vulgar fractions, we thence immediately deduce that of decimals, and the manner of performing the multiplication and division of them; but we can also arrive at this theory directly by the following considerations.

Let us first suppose only the multiplicand to have decimal figures. If the comma be taken away, it will become ten, a hundred, a thousand, &c. times greater, according to the number of decimal figures; and in this case the product given by multiplication will be a like number of times greater than the one required; the latter will then be obtained by dividing the former by ten, a hundred, a thousand, &c. which may be done by separating on the right (90) as many decimal figures, as there are in the multiplicand.

If, for instance, 34,137 were to be multiplied by 9, we must first find the product of 34137 by 9, which will be 307233; and, since taking away the comma renders the multiplicand a thousand times greater, we must divide this product by a thousand, or separate by a comma, its three last figures on the right; we shall thus have 307,233.

In general, *to multiply, by a whole number, a number accompanied by decimals, the comma must be taken away from the multiplicand, and as many figures separated for decimals, on the right of the product, as are contained in the multiplicand.*

Examples for practice.

Multiply 231,415 by 8. *Ans.* 1851,320.
Multiply 32,1509 by 15. *Ans.* 482,2635.
Multiply ,840 by 840. *Ans.* 705,600.
Multiply 1,236 by 13. *Ans.* 16,068.

92. When the multiplier contains decimal figures by suppressing the comma, it is made ten, a hundred, a thousand, &c. times greater according to the number of decimal figures. If used in this state, it will evidently give a product, ten, a hundred, a thousand, &c. times greater than that which is required, and consequently the true product will be obtained by dividing by one of these numbers, that is, by separating, on the right of it, as many decimal figures as there are in the multiplier, or by removing the comma a like number of places towards the left (90), in case it previously existed in the product on account of decimals in the multiplicand. For instance, let 172,84 be multiplied by 36,003; taking away the comma in the multiplier only, we shall have, according to the preceding article, the product 6222758,52; but, the multiplier being rendered a thousand times too great, we must divide this product by a thousand, or remove the comma three places towards the left, and the required product will then be 6222,75852, in which there must necessarily be as many decimal figures as there are in both multiplicand and multiplier.

In general, *to multiply one by the other, two numbers accompanied by decimals, the comma must be taken away from both, and as many figures separated for decimals, on the right of the product, as there are in both the factors.*

In some cases it is necessary to put one or more ciphers on the left of the product, to give the number of decimal figures required by the above rule. If, for example, 0,624 be multiplied by 0,003; in forming at first the product of 624 by 3, we shall have the number 1872, containing but 4 figures, and as 6 figures must be separated for decimals, it cannot be done except by placing on the left three ciphers, one of which must occupy the place of units, which will make 0,001872.

Examples for practice.

Multiply 223,86 by 2,500. *Ans.* 559,65000.
Multiply 35,640 by 26,18. *Ans.* 933,05520.
Multiply 8,4960 by 2,618. *Ans.* 22,2425280.
Multiply ,5236 by ,2808. *Ans.* 0,14702688.
Multiply ,11785 by ,27. *Ans.* 0,0318195.

93. It is evident (36), that the quotient of two numbers does not depend on the absolute magnitude of their units, provided that this be the same in each; if then, it be required to divide 451,49 by 13, we should observe that the former amounts to 45149 hundredths, and the latter to 1300 hundredths, and that these last numbers ought to give the same quotient, as if they expressed units. We shall thus be led to suppress the point in the first number, and to put two ciphers at the end of the second, and then we shall only have to divide 45149 by 1300, the quotient of which division will be $34\frac{949}{1300}$.

Hence we conclude, that, *to divide, by a whole number, a number accompanied by decimal figures, the comma in the dividend must be taken away, and as many ciphers placed at the end of the divisor, as the dividend contains decimal figures, and no alteration in the quotient will be necessary.*

94. When both dividend and divisor are accompanied by decimal figures, we must, before taking away the comma, reduce them to decimals of the same order, by placing at the end of that number, which has the fewest decimal figures, as many ciphers as will make it terminate at the same place of decimals as the other, because then the suppression of the comma renders both the same number of times greater.

For instance, let 315,432 be divided by 23,4, this last must be changed into 23,400, and then 315432 must be divided by 23400; the quotient will be $13\frac{11232}{23400}$.

Thus, *to divide one by the other, two numbers accompanied by decimal figures, the number of decimal figures in the divisor and dividend must be made equal, by annexing to the one, that has the least, as many ciphers as are necessary; the point must then be suppressed in each, and the quotient will require no alteration.*

95. As we have recourse to decimals only to avoid the neces-

sity of employing vulgar fractions, it is natural to make use of decimals for approximating quotients that cannot be obtained exactly, which is done by converting the remainder into tenths, hundredths, thousandths, &c. so that it may contain the divisor; as may be seen in the following example;

	45149	1300
	3900	34,73
	6149	
	5200	
Remainder	949	
tenths	9490	
	9100	
hundredths	3900	
	3900	
	0	

When we come to the remainder 949, we annex a cipher in order to multiply it by ten, or to convert it into tenths; thus forming a new partial dividend, which contains 9490 tenths and gives for a quotient 7 tenths, which we put on the right of the units, after a comma. There still remains 390 tenths, which we reduce to hundredths by the addition of another cipher, and form a second dividend, which contains 3900 hundredths, and gives a quotient, 3 hundredths, which we place after the tenths. Here the operation terminates, and we have for the exact result 34,73 hundredths. If a third remainder had been left, we might have continued the operation, by converting this remainder into thousandths, and so on, in the same manner, until we came to an exact quotient, or to a remainder composed of parts so small, that we might have considered them of no importance.

It is evident, that we must always put a comma, as in the above example, after the whole units in the quotient, to distinguish them from the decimal figures, the number of which must be equal to that of the ciphers successively written after the remainders*.

* The problem above performed with respect to decimals, is only

Examples for practice.

Divide 6345,925	by 54,23.	*Ans.* 117,018 &c.
Divide 5673,21	by 23,0.	*Ans.* 246,660 &c.
Divide 84329907	by 627,1.	*Ans.* 134476,10 &c.
Divide 27845,96	by 9,8732.	*Ans.* 2820,3581 &c.
Divide 200,5	by 231.	*Ans.* 0,0867 &c.
Divide 10,0	by 563,0.	*Ans.* 0,00177 &c.
Divide 513,2	by 0,057.	*Ans.* 9003,50 &c.
Divide 7,25406	by 957.	*Ans.* 0,00758 &c.
Divide 0,00078759	by 0,525.	*Ans.* 0,00150 &c.
Divide 14	by 365.	*Ans.* 0,038356 &c.

96. The numerator of a fraction, being converted into decimal parts, can be divided by the denominator as in the preceding examples, and by this means the fraction will be converted into decimals. Let the fraction, for example, be $\frac{1}{8}$, the operation is performed thus;

```
  1 | 8
 ---|------
 10 | 0,125
  8 |
 ---
  20
  16
  --
  40
  40
  --
   0
```

Again, let the fraction be $\frac{4}{797}$; the numerator must be converted into thousandths before the division can begin.

---

a particular case of the following more general one; *To find the value of the quotient of a division, in fractions of a given denomination;* to do this, we convert the dividend into a fraction of the same denomination by multiplying it by the given denominator. Thus, in order to find in fifteenths the value of the quotient of 7 by 3, we should multiply 7 by 15, and divide the product, 105, by 3, which gives thirty-five fifteenths, or $\frac{35}{15}$ for the quotient required.

```
4    | 797    *
-----|-------------
4000 | 0,005018 &c.
3985 |
-----
  1500
   797
  ----
  7030
  6376
  ----
   654
```

Examples for practice.

Reduce $\frac{3}{4}$ to a decimal fraction. *Ans.* 0,75.

Reduce $\frac{1}{2}$ to a decimal fraction. *Ans.* 0,5.

Reduce $\frac{5}{70}$ to a decimal fraction. *Ans.* 0,0714285 &c.

Reduce $\frac{5}{100}$ to a decimal fraction. *Ans.* 0,05.

Reduce $\frac{3}{9}$ to a decimal fraction. *Ans.* 0,333 &c.

97. However far we may continue the second division, exhibited above, we shall never obtain an exact quotient, because the fraction $\frac{4}{797}$ cannot, like $\frac{1}{8}$, be exactly expressed by decimals.

The difference in the two cases arises from this, that the denominator of a fraction, which does not divide its numerator, cannot give an exact quotient, except it will divide one of the numbers 10, 100, 1000, &c. by which its numerator is successively multiplied, because it is a principal, which will be found demonstrated in Algebra, that no number will divide a product, except its factors will divide those of the product; now the numbers 10, 100, 1000, &c. being all formed from 10, the factors of which are 2 and 5, they cannot be divided except by

---

* It may also be proposed to convert a given fraction into a fraction of another denomination, but smaller than the first, for instance, $\frac{3}{4}$ into seventeenths, which will be done by multiplying 3 by 17 and dividing the product by 4. In this manner we find $\frac{51}{4}$ seventeenths, or $\frac{12}{17}$ and $\frac{3}{4}$ of a seventeenth; but $\frac{3}{4}$ of $\frac{1}{17}$ is equivalent to $\frac{3}{68}$. The result then, $\frac{12}{17}$, is equal to $\frac{3}{4}$, wanting $\frac{3}{68}$.

This operation and that of the preceding note depend on the same principle, as the corresponding operation for decimal fractions.

numbers formed from these same factors; 8 is among these, being the product of 2 by 2 by 2.

Fractions, the value of which cannot be exactly found by decimals, present in their approximate expression, when it has been carried sufficiently far, a character which serves to denote them; this is the periodical return of the same figures.

If we convert the fraction $\frac{12}{37}$ into decimals, we shall find it 0,324324 . . . . ., and the figures 3, 2, 4, will always return in the same order, without the operation ever coming to an end.

Indeed, as there can be no remainder in these successive divisions except one of the series of whole numbers 1, 2, 3, &c. up to the divisor, it necessarily happens, that, when the number of divisions exceeds that of this series, we must fall again upon some one of the preceding remainders, and consequently the partial dividends will return in the same order. In the above example three divisions are sufficient to cause the return of the same figures; but six are necessary for the fraction $\frac{1}{7}$, because in this case we find, for remainders, the six numbers which are below 7, and the result is 0,1428571 . . . . The fraction $\frac{1}{3}$ leads only to 0,3333 . . . . .

98. The fractions, which have for a denominator any numbers of 9s, have no significant figure in their periods except 1;

$\frac{1}{9}$ gives 0,11111 . . .
$\frac{1}{99}$ 0,010101 . . . . .
$\frac{1}{999}$ 0,001001001 . . . . .

and so with the others, because each partial division of the numbers 10, 100, 1000, &c. always leaves unity for the remainder.

Availing ourselves of this remark, we pass easily from a periodical decimal, to the vulgar fraction from which it is derived. We see, for example, that 0,33333 . . . . . amounts to the same as 0,11111 . . . . . multiplied by 3, and as this last decimal is the development of $\frac{1}{9}$, or $\frac{1}{9}$ reduced to a decimal, we conclude, that the former is the development of $\frac{1}{9}$ multiplied by 3, or $\frac{3}{9}$, or lastly, $\frac{1}{3}$.

When the period of the fraction under consideration consists of two figures, we compare it with the development of $\frac{1}{99}$, and with that of $\frac{1}{999}$, when the period contains three figures, and so on.

If we had, for example, 0,324324, it is plain that this fraction may be formed by multiplying 0,001001 . . . . . by 324; if we multiply then $\frac{1}{999}$, of which 0,001001 is the development, by 324, we obtain $\frac{324}{999}$, and dividing each term of this result by 27, we come back again to the fraction $\frac{12}{37}$.

In general, *the vulgar fraction, from which a decimal fraction arises, is formed by writing, as a denominator, under the number, which expresses one period, as many 9s, as there are figures in the period.*

If the period of the fraction does not commence with the first decimal figure, we can for a moment change the place of the point, and put it immediately before the first figure of the period, and, beginning with this figure, find the value of the fraction, as if those figures on the left were units; nothing then will be necessary except to divide the result by 10, 100, 1000, &c. according to the number of places the point was moved towards the right.

For instance, the fraction 0,324141, is first to be written 32,4141; the part 0,4141 being equivalent to $\frac{41}{99}$, we shall have $32\frac{41}{99}$. which is to be divided by 100, because the point was moved two places towards the left; it will consequently become $\frac{32}{100}$ and $\frac{41}{9900}$, or by reducing the two parts to the same denominator, and adding them, $\frac{3209}{9900}$, a fraction, which will reproduce the given expression.

*Examples for practice**.

Reduce $0,\dot{1}\dot{8}$ to the form of a vulgar fraction. *Ans.* $\frac{2}{11}$

Reduce $0,\dot{7}\dot{2}$ to the form of a vulgar fraction. *Ans.* $\frac{8}{11}$.

Reduce $0,8\dot{3}$ to the form of a vulgar fraction. *Ans.* $\frac{5}{6}$.

Reduce $0,2\dot{4}1\dot{8}$ to the form of a vulgar fraction. *Ans.* $\frac{1208}{4995}$.

Reduce $0,27\dot{5}46\dot{3}$ to the form of a vulgar fraction.
Ans. $\frac{22953}{83325}$.

Reduce $0,91\dot{6}$ to the form of a vulgar fraction. *Ans.* $\frac{11}{12}$.

* In these examples, the better to distinguish the period, a point is placed over it, if it be a single figure, and over the first and last figure, if it consist of more than one.

To form a correct idea of the nature of these fractions it is sufficient to consider the fraction 0,999. In trying to discover its original value we find that it answers to 9 divided by 9, that is to unity; nevertheless, at whatever number of figures we stop in its expression, it will never make an unit. If we stop at the first figure, it wants $\frac{1}{10}$ of an unit; if at the second, it wants $\frac{1}{100}$; if at the third, it wants $\frac{1}{1000}$, and so on; so that we can arrive as near to unity as we please, but can never reach it. Unity then in this case is nothing but a *limit*, to which 0,999 . . . . . . continually approaches the nearer the more figures it has.

99. The preceding part of this work contains all the rules absolutely essential to the arithmetic of abstract numbers, but to apply them to the uses of society it is necessary to know the different kinds of units, which are used to compare together, or ascertain the value of quantities, under whatever form they may present themselves. These units, which are the measures in use, have varied with times and places, and their connexion has been formed only by degrees, accordingly as necessity and the progress of the arts and sciences have required greater exactness in the valuation of substances, and the construction of instruments.

TABLES OF COIN, WEIGHT, AND MEASURE.

Denominations of Federal money, as determined by an act of Congress, Aug. 8, 1786†.

		Marked
10 mills make	one cent	c.
10 cents	one dime	d.
10 dimes	one dollar	$.
10 dollars	one eagle	E.

† The coins of federal money are two of gold, four of silver, and two of copper. The gold coins are an *eagle* and *half-eagle;* the silver, a *dollar, half-dollar, double dime,* and *dime;* and the copper a *cent* and *half-cent.* The standard for gold and silver is eleven parts fine and one part alloy. The weight of fine gold in the eagle is 246,268 grains; of fine silver in the dollar, 375,64 grains; of copper

English Money.

4 farthing make	1 penny	£ denotes		pounds.
12 pence	1 shilling	*s*		shillings.
20 shillings	1 pound	*d*		pence.
		q		quarters or farthings.

TROY WEIGHT.

24 grains make	1 penny-weight, marked	grs. dwt.
20 dwt.	1 ounce,	oz.
12 oz.	1 pound,	lb.

By this weight are weighed jewels, gold, silver, corn, bread and liquors.

APOTHECARIES' WEIGHT.

20 grains make	1 scruple, marked	gr. sc.
3 sc.	1 dram	dr. or ʒ.
8 dr.	1 ounce	oz. or ℥.
12 oz.	1 pound	lb.

Apothecaries use this weight in compounding their medicines;

---

in 100 cents, $2\frac{1}{4}$ lb. avoirdupois. The fine gold in the half-eagle is half the weight of that in the eagle; the fine silver in the half-dollar, half the weight of that in the dollar, &c. The denominations less than a dollar are expressive of their values; thus, *mill* is an abbreviation of *mille*, a thousand, for 1000 mills are equal to 1 dollar; *cent*, of *centum*, a hundred, for 100 cents are equal to 1 dollar; a *dime* is the French of *tithe*, the tenth part, for 10 dimes are equal to 1 dollar.

The mint-price of uncoined gold, 11 parts being fine and 1 part alloy, is 209 dollars, 7 dimes, and 7 cents per lb. Troy weight; and the mint-price of uncoined silver, 11 parts being fine and 1 part alloy, is 9 dollars, 9 dimes, and 2 cents, per lb. Troy.

In practical treatises on arithmetic, may be found rules for reducing the Federal Coin, the currencies of the several United States, and those of foreign countries, each to the *par* of all the others. It may be sufficient here to observe respecting the currencies of the several states, that a dollar is equal to 6s. in New-England and Virginia; 8s. in New York and North Carolina; 7s. 6d. in New-Jersey, Pennsylvania, Delaware, and Maryland; and 4s. 8d. in South Carolina and Georgia.

but they buy and sell their drugs by Avoirdupois weight. Apothecaries' is the same as Troy weight, having only some different divisions.

AVOIRDUPOIS WEIGHT.

16 drams make	1 ounce, marked	dr. oz.
16 ounces	1 pound	lb.
28 lb.	1 quarter	qr.
4 quarters	1 hundred weight	cwt.
20 cwt.	1 ton	T.

By this weight are weighed all things of a coarse or drossy nature; such as butter, cheese, flesh, grocery wares, and all metals, except gold and silver.

DRY MEASURE.

		Marked			Marked
2 pints make	1 quart pts.	qts.	8 bushels	1 quarter	qr.
2 quarts	1 pottle	pot.	5 quarters	1 wey *or* load	wey
2 pottles	1 gallon	gal.	4 bushels	1 coom *or* carnock	co.
2 gallons	1 peck	pe.	2 cooms	a seam *or* quarter.	
4 pecks	1 bushel.	bu.	6 seams	1 wey.	
2 bushels	1 strike.	str.	$1\frac{2}{3}$ weys	1 last	L.

The diameter of a Winchester bushel is $18\frac{1}{2}$ inches, and its depth 8 inches.—And one gallon by dry measure contains $268\frac{4}{5}$ cubic inches.

By this measure, salt, lead, ore, oysters, corn, and other dry goods are measured.

ALE AND BEER MEASURE.

		Marked			Marked
2 pints make	1 quart pts.	qts.	2 firkins	1 kilderkin	kil.
4 quarts	1 gallon	gal.	2 kilderkins	1 barrel	bar.
8 gallons	1 firkin of Ale	fir.	3 kilderkins	1 hogshead	hhd.
9 gallons	1 firkin of Beer	fir.	3 barrels	1 butt	butt.

The ale gallon contains 282 cubic inches. In London the ale firkin contains 8 gallons, and the beer firkin 9; other measures being in the same proportion.

WINE MEASURE.

			Marked				Marked
2 pints	make	1 quart pts.	qts.	2 hogsheads		1 pipe *or* butt	p. *or* b.
4 quarts		1 gallon	gal.	2 pipes		1 tun	T.
42 gallons		1 tierce	tier.	18 gallons		1 runlet	run.
63 gallons		1 hogshead	hhd.	31½ gallons		1 barrel.	bar.
84 gallons		1 puncheon	pun.				

By this measure, brandy, spirits, perry, cider, mead, vinegar, and oil are measured.

231 cubic inches make a gallon, and 10 gallons make an anchor.

CLOTH MEASURE.

			Marked			Marked
$2\frac{1}{4}$ inches	make	1 nail	nls.	3 qrs.	1 ell Flemish	Ell Fl.
4 nails		1 quarter	qrs.	5 qrs.	1 ell English	Ell Eng.
4 quarters		1 yard	yds.	6 qrs.	1 ell French.	Ell Fr.

LONG MEASURE.

			Marked			Marked
3 barley corns	make	1 inch	bar. c. in.	60 geographical miles, *or* $69\frac{1}{5}$ statute miles	1 degree nearly	deg. *or* °
12 inches		1 foot	ft.	360 degrees	the circumference of the earth.	
3 feet		1 yard	yd.	*Also,* 4 inches	make 1 hand.	
6 feet		1 fathom	fath.	5 feet	1 geometrical pace.	
$5\frac{1}{2}$ yards		1 pole	pol.	6 points	1 line	
40 poles		1 furlong	fur.	12 lines	1 inch.	
8 furlongs		1 mile	mls.			
3 miles		1 league	l.			

TIME.

			Marked			Marked
60 seconds	make	1 minute	s. *or* ″ m. *or* ′	4 weeks	1 month	m.
60 minutes		1 hour	h. *or* °	13 months, 1 day, and 6 hours, *or* 365 days and 6 hours,	1 Julian year.	Y.
24 hours		1 day	d.			
7 days		1 week	w.			

100. It is evident, that if the several denominations of money, weight and measure proceeded in a decimal ratio, the fundamental operations might be performed upon these, as upon abstract numbers. This may be shown by a few examples in

Federal Money. If it were required to find the sum of \$46,85 and \$256,371, we should place the numbers of the same denomination in the same column, and add them together as in whole numbers; thus,

```
  4685
256371
------
303221
```

and the answer may be read off in either or all of the denominations; we may say 30 eagles 3 dollars 22 cents 1 mill, or 303 dollars 221 thousandths, or 30322 cents and 1 tenth, or 303221 mills. It is usual to consider the dollars as whole numbers, and the following denominations as decimals. The operation then becomes the same as for decimals.

Examples.

Add	\$34,123	Add	\$456,78
	1,178		49,83
	78,001		0,22
	61,789		7854,394
Sum	\$175,091	Sum	\$8361,224

From	\$542,76	From	\$527,839
Subtract	239,481	Subtract	22,94
Rem.	303,269	Rem.	504,899

Multiply \$6,347 by \$4,532. *Ans.* \$28,764604.
Divide \$28,764604 by \$4,532. *Ans.* \$6,347.
Divide \$20 by \$2000. *Ans.* \$0,01.

Reduction.

101. When the different denominations do not proceed in a decimal ratio, they may all be *reduced* to one denomination, and then the fundamental operations may be performed upon this, as upon an abstract number. If, for example, the sum to be operated upon were £4 15s. 9d, this may easily be expressed in

pence. As 1 pound is 20 shillings, 4 pounds will be 4 times 20, or 80 shillings. If to this we add the 15s. we shall have 95s. 9d. equivalent to the above. But as 1 shilling is equal to 12 pence, 95s. will be equal to 95 times 12 or 1140 pence. Adding 9 to this, we shall have 1149 pence as an equivalent expression for £4 15s. 9d. We may now make use of this number as if it had no relation to money or any thing else ; and the result obtained may be converted again into the different denominations by reversing the process above pursued. If it were proposed to multiply this sum by another number, 37, for instance, we should find the product of these two numbers in the usual way ; thus,

```
 1149
   37
 ----
 8043
3447
-----
42513
```

42513 is therefore, equal to 37 times £4 15s. 9d. expressed in pence ; to find the number of pounds and shillings contained in this, we first obtain the number of shillings by dividing it by 12, which gives 3542, and then the number of pounds by dividing this last by 20 ; thus,

```
42513 | 12            354,2 | 20
 65   |-----          15    |-----
  51  | 3542           14   | 177
   33 |                  2
    9 |
------
```

42513 pence then is equal to 3542 shillings, or to 175 pounds 15 shillings. Whence 37 times £4 15s. 9d. is equal to £177 2s. 9d.

It may be remarked, that *shillings are converted into pounds by separating the right hand figure and dividing those on the left by 2,* prefixing the remainder, if there be one, to the figure separated for the entire shillings, that remain. This amounts to dividing, first, by 10 (90), and then that quotient by 2. If 10 shillings made a pound dividing by 10 would give the number of pounds, but as 10 shillings is only half a pound, half this number will be the number of pounds.

By a method similar to that above given, we reduce other denominations of money and the different denominations of the several weights and measures to the lowest respectively. If it were required to find how many grains there are in 2lb. 4oz. 17dwt. 5grs. Troy, we should proceed thus,

```
  lb.   oz.   dwt.   grs.
    2    4     17     5
   12
 ----
   24
    4
 ----
   28
   20
 ----
  560
   17
 ----
  577
   24
 ----
 2308
1154
 -----
13848
    5
 -----
Ans. 13853
```

By dividing 13853 by 24, and the quotient thence arising by 20, and this second quotient by 12, we shall evidently obtain the number of pounds, ounces, pennyweights and grains in 13853 grains. The operation may be seen below.

```
13853 | 24
120   |----
----    577 | 20
 185    40  |----
 168    ---    28 | 12
 ---    177    24 |----
  173   160    --    2
  168   ---     4
  ---    17
    5
```

Result 2 lb. 4 oz. 17 dwt. 5 gr.

These examples will be sufficient to establish the following general rules, namely;

To reduce a compound number to the lowest denomination contained in it, multiply the highest by so many as one of this denomination makes of the next lower, and to the product add the number belonging to the next lower; then multiply this sum by so many as one of this makes of the third lower, adding the number of the lower, as before, and so on through the whole, and the last sum will be the number required.

To reduce a number from a lower denomination to a higher, divide by so many as it takes of this lower denomination to make one of the higher, and the quotient will be the number of the higher; which may be further reduced in the same manner if there are still higher denominations, and the last quotient together with the several remainders will be equivalent to the number to be reduced.

Examples for practice.

In 59lb. 13dwt. 5gr. how many grains? *Ans.* 340157.

In 8012131 grains how many pounds, &c.?

Ans. 139lb. 11oz. 18dwt. 19gr.

In 121l. 0s. 9½d how many half pence? *Ans.* 58099.

In 58099 half pence how many pounds &c.? *Ans.* 121*l.* 0s. 9½d.

In 48 guineas at 28s. each how many 4½d. pence?

Ans. 3584.

In one year of 365d. 5h 48′ 48″ how many seconds?

Ans. 31556928.

102. When we have occasion to make use of a number consisting of several denominations as an abstract number, instead of reducing the several parts to the lowest denomination contained in it, we may reduce all the lower denominations to a fraction of the highest. Taking the sum before used, namely, 4l. 15s. 9d. we reduce the lower denominations, to the higher, as in the last article by division. The number of pence 9, or $\frac{9}{1}$, is divided by 12, by multiplying the denominator by this number (54), we have thus, $\frac{9}{12}$s. which being added to 15s. or $\frac{180}{12}$s. the whole number being reduced to the form of a fraction of the same denominator, we have $\frac{180}{12}$ and $\frac{9}{12}$, which being added, make $\frac{189}{12}$. This is further reduced to pounds by dividing it by 20,

that is, by multiplying the denominator by 20 (54), which gives $\frac{189}{240}$. Whence £4. 15s. 9d. is equal to £$4\frac{189}{240}$, or £$\frac{1169}{240}$. This may now be used like any other fraction and the value of the result found in the different denominations. If we multiply it by 37, we shall have £$\frac{42513}{240}$, or £$177\frac{33}{240}$; and £$\frac{33}{240}$, reduced to shillings by multiplying the numerator by 20, or dividing the denominator by this number, gives $\frac{33}{12}$s. or 2 $\frac{9}{12}$s. or 2s. 9d.

From the above example, we may deduce the following general rules, namely,

To reduce the several parts of a compound number to a fraction of the highest denomination contained in it, make the lowest term the numerator of a fraction, having for its denominator the number which it takes of this denomination to make one of the next higher and add to this the next term reduced to a fraction of the same denomination, then multiply the denominator of this sum by so many as make one of the next denomination, and so on through all the terms, and the last sum will be the fraction required†.

To find the value of a fraction of a higher denomination in terms of a lower, multiply the numerator of the fraction by so many as make one of the lower denomination, and divide the product by the denominator, and the quotient will be the entire number of this denomination, the fractional part of which may be still further reduced in the same manner.

To reduce 2w. 1d. 6h. to the fraction of a month.

6h. is $\frac{6}{24}$ of a day, and being added to one day, or $\frac{24}{24}$d. gives $\frac{30}{24}$d. the denominator of which being multiplied by 7, it becomes $\frac{30}{168}$w. If we now multiply the denominator of this by 4, we shall have $\frac{30}{672}$ of a month as an equivalent expression for 2w. 1d. 6h.

To find the value of $\frac{5}{7}$ of a mile in furlongs, poles, &c.

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† It will often be found more convenient to reduce the several parts of the compound number to the lowest denomination, as by the preceding article, for a numerator, and to take for the denominator so many of this denomination as it takes to make one of that, to which the expression is to be reduced; thus 4l. 15s. 9d. being 1149d. is equal to $\frac{1149}{960}$l. because 1d. is $\frac{1}{960}$l.

```
   5
   8
 ----
  40 | 7
  35 |---
 ----| 5
   5
  40
 ----  7
 200 ----
  14  28
 ----
  60
  56
 ----
   4
   5½     7
 ----|  ----
  22 |  3 1/7
  21 |
 ----
   1
```

Ans. 5fur. 28pls. $3\frac{1}{7}$yds.

Reduce 13s. 6d. 2q. to the fraction of a pound.

Ans. £$\frac{650}{960}$, or £$\frac{65}{96}$.

Reduce 6fur. 26pls. 3yds. 2ft. to the fraction of a mile.

Ans. $\frac{4400}{5280}$, or $\frac{5}{6}$.

Reduce 7oz. 4pwt. to the fraction of a pound, Troy. *Ans.* $\frac{3}{5}$.

What part of a mile is 6fur. 16pls.? *Ans.* $\frac{4}{5}$.

What part of a hogshead is 9 gallons? *Ans.* $\frac{1}{7}$.

What part of a day is $\frac{3}{13}$ of a month? *Ans.* $\frac{84}{13}$.

What part of a penny is $\frac{1}{18}$ of a pound? *Ans.* $\frac{40}{3}$.

What part of a cwt. is $\frac{6}{7}$ of a pound, Avoirdupois? *Ans.* $\frac{3}{392}$.

What part of a pound is $\frac{2}{3}$ of a farthing? *Ans.* $\frac{1}{1440}$.

What is the value of $\frac{3}{5}$ of a pound, Troy? *Ans.* 7oz. 4dwt.

What is the value of $\frac{4}{7}$ of pound, Avoirdupois?

Ans. 9oz. $2\frac{2}{7}$dr.

What is the value of $\frac{7}{9}$ of a cwt.? *Ans.* 3qrs. 3lb. 1oz. $12\frac{4}{9}$dr.

What is the value of $\frac{3}{17}$ of a mile?

Ans. 1fur. 16pls. 2yds. 1ft. $9\frac{3}{17}$in.

What is the value of $\frac{7}{13}$ of day? *Ans.* 12h. 55′ $23\frac{1}{13}$″.

The several parts of a compound number may also be reduced to the form of a decimal fraction of the highest denomination contained in it, by first finding the value of the expression in a vulgar fraction as in the last article, and then reducing this to a decimal, or more conveniently by changing the terms to be reduced into decimal parts, and dividing the numerator instead of multiplying the denominator by the numbers successively employed in raising them to the required denomination.

If we take the sum already used, namely, £4 15s. 9d. the pence, 9, may be written $\frac{90}{10}$, or $\frac{900}{100}$, the numerator of which admits of being divided by 12 without a remainder. It is thus reduced to shillings and becomes $\frac{75}{100}$s. or 0,75s. which added to the 15s. makes 15,75s. or reducing the 15 to the same denomination, $\frac{1575}{100}$, or $\frac{157500}{10000}$; and this is reduced to pounds, by dividing it by 20, the result of which is $\frac{7875}{10000}$, or 0,7875. 4l. 15s. 9d. therefore may be expressed in one denomination, thus 4,7875l. and in this state, it may be used like any other number consisting of an entire and fractional part. If it be multiplied by 37 we shall have for the product 177,1375l. This decimal of a pound may be reduced to shillings and pence, by reversing the above process, or by multiplying successively by 20 and then by 12.

$$\begin{array}{r} 0,1375 \\ 20 \\ \hline 2,7500 \\ 12 \\ \hline 9,0000 \end{array}$$

The product therefore of 4l. 15s. 9d. by 37 is 177l. 2s. 9d. as before obtained.

The operation, just explained, admits of a more convenient disposition, as in the following example.

To reduce 19s. 3d. 3q. to the decimal of a pound.

$$\begin{array}{r|l} 4 & 3,00 \\ 12 & 3,75000 \\ 20 & 19,312500 \\ \hline & 0,965625 \end{array}$$

Proceeding as before, we reduce the farthings, 3, considered as $\frac{300}{100}$q. to hundredths of a penny by dividing by the figure on the left, 4, and place the quotient, 75, as a decimal on the right of the pence; we then take this sum, considered as $\frac{375}{100}$d. or $\frac{3750}{1000}$d. that is, annexing as many ciphers as may be necessary, and divide it by 12, which brings it into decimals of a shilling. Lastly, the shillings and parts of a shilling, 19,3125s. considered as $\frac{19312500}{1000000}$s. are reduced to decimals of a pound by dividing by 20, which gives the result above found.

We may proceed in a similar manner with other denominations of money and with those of the several weights and measures. One example in these will suffice as an illustration of the method.

To reduce 17pls. 1ft. 6in. to the decimal of a mile.

12	6
3	1,5
320	17,5
	0,00994318 &c.

The decimal in this, as in many other cases, becomes periodical (97).

From what has been said, the following rules are sufficiently evident. *To reduce a number from a lower denomination to the decimal of a higher, we first change it, or suppose it to be changed into a fraction, having* 10, *or some multiple of* 10, *for its denominator, and divide the numerator by so many as make one of this higher denomination, and the quotient is the required decimal; which together with the whole number of this denomination, may again be converted into a fraction, having* 10 *or a multiple of* 10 *for its denominator, and thus by division be reduced to a still higher name, and so on.*

Also, *to reduce a decimal of a higher denomination to a lower, we multiply it by so many as one makes of this lower, and those figures which remain on the left of the comma, when the proper number are separated for decimals* (91), *will constitute the whole number of this denomination, the decimal part of which may be still further reduced, if there be lower denominations, by multiplying it by the number which one makes of the next denomination, and so on.*

It may be proper to add in this place, that shillings, pence and farthings may readily be converted into the fraction of a pound, and the fraction of a pound reduced to shillings, pence and farthings, without having recourse to the above rules. As shillings are so many twentieths of a pound, by dividing any given number of shillings by 2, we convert them into decimals of a pound, thus, 15s. which may be written $\frac{15}{20}$l. or $\frac{150}{200}$l. being divided by 2 give 75 hundredths, or 0,75 of a pound. Also, as farthings are so many 960ths of a pound, one pound being equal to 960 farthings, the pence converted into farthings and united with those of this denomination, may be written as so many 960ths of a pound. If now we increase the numerator and denominator one twenty fourth part, we shall convert the denominator into thousandths, and the numerator will become a decimal.

Whence, *to convert shillings, pence and farthings, into the decimal of a pound, divide the shillings by* 2, *adding a cipher when necessary, and let the quotient occupy the first place, or first and second, if there be two figures, and let the farthings, contained in the pence and farthings, be considered as so many thousandths, increasing the number by one, when the number is nearer* 24 *than* 0, *and by* 2, *when it is nearer* 48 *than* 24, *and so on.*

Thus, to reduce 15s. 9d. to the decimal of a pound, we have,

$$\begin{array}{r} 0,75 \\ 37 \\ \hline 0,787 \end{array}$$

This result, it will be remarked, is not exactly the same as that obtained by the other method; the reason is, that we have increased the number of farthings, 36, by only one, whereas, allowing one for every 24, we ought to have increased it one and a half. Adding therefore, a half, or 5 units of the next lower order, we shall have 0,7875, as before.

On the other hand, *the decimal of a pound is converted into the lower denominations, or its value is found in shillings, pence, and farthings, by doubling the first figure for shillings, increasing it by one, when the second figure is* 5, *or more than* 5, *and considering what remains in the second and third places, as farthings, after having diminished them one for every* 24.

In addition to the rules that have been given, it may be observed, that in those cases, where it is required to reduce a number from one denomination to another, when the two denominations are not commensurable, or when one will not exactly divide the other, it will be found most convenient, as a general rule, to reduce the one, or both, when it is necessary, to parts so small, that a certain number of the one will exactly make a unit of the other. If it were required, for instance, to reduce pounds to dollars, as a pound does not contain an exact number of dollars without a fraction, we first convert the pounds into shillings, and then, as a certain number of shillings make a dollar, by dividing the shillings by this number, we shall find the number of dollars required. A similar method may be pursued in other cases of a like nature, as may be seen in the following examples.

In 178 guineas at 28s. each, how many crowns at 6s. 8d.?

6s. 8d.	178	5980,8 | 80
12	28	48 | 747
80d.	1424	
	356	
	4984	
	12	
	59803	

Ans. 747 crowns and 4 shillings†.

In this case, I reduce both the guineas and the crown to pence, and then divide the former result by the latter. In dividing by 80, I first separate one figure on the right of the dividend for a decimal, which is the same as dividing it by 10, and then divide the figures on the left, or the quotient, by 8 (47), joining what remains as tens to the figure separated, to form the entire remainder, which is reduced back to the original denomination.

To reduce 137 five franc pieces to pounds, shillings, &c. the franc being valued at $0,1796.

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† Questions of this kind may often be conveniently performed by fractions; thus, 178 guineas, or 4984s. divided by 6s. 8d. or $6\frac{2}{3}$s. or reducing the whole number to the form of a fraction, $\frac{20}{3}$s. becomes $\frac{4984}{1}$ multiplied by $\frac{3}{20}$ (74), or $\frac{14952}{20}$, or $\frac{1495,2}{2}$, which is equal to $747\frac{12}{20}$; and $\frac{12}{20}$, or $\frac{3}{5}$, of 6s. 8d. is 3 times $\frac{1}{5}$ of 80d. or 48d. or 4s.

0,1796	73,8156 | 20
5	36,9078
0,8980	20
137	18,1560
6286	12
2694	1,8720
898	4
123,026	3,4880
6	
738,156	

Ans. 36l. 18s. 1d. $3\frac{1}{2}$q. nearly.

Examples for practice.

Reduce 7s. $9\frac{3}{4}$d. to the decimal of a pound. *Ans.* 0,4025.

Reduce 3qrs. 2na. to the decimal of a yard. *Ans.* 0,875.

Find the value of 0,85251l. in shillings, pence, &c.

Ans. 17s. 0d. $2\frac{1}{2}$q. nearly.

Reduce 241l. 18s. 9d. to federal money. *Ans.* $967,75.

Find the value of 0,42857 of a month.

Ans. 1w. 4d. 23h. 59′ 56″.

Required the circumference of the earth in English statute miles, a degree being estimated at 57008 toises†.

Ans. 24855,488.

We have given rules for reducing a compound number from one denomination to another, as we shall have frequent occasion in what follows for making these reductions. They are not, however necessary, except in particular cases, previously to performing the fundamental operations. The several denominations of a compound number may be regarded like the different orders of units in a simple one, that is, the number or numbers of each denomination may be made the subject of a distinct operation, the result of which, being reduced when necessary, may be united to the next, and so on through all the denominations.

† A toise or French fathom is equal to 6 French feet, and a French foot is equal to 12,789$\dot{3}$ English inches.

Addition of Compound Numbers.

103. The addition of compound numbers depends on the same principles as that of simple numbers, the object being simply to unite parts of the same denomination, and when a number of these are found, sufficient to form one, or more than one of a higher, these last are retained to be united to others of the same denomination in the given numbers; as in simple addition the tens are carried from one column to the next column on the left. *We must, then, place the compound numbers, that are to be added, in such a manner, that their units, or parts of the same name, may stand under each other; we must then find separately the sum of each column, always recollecting how many parts of each denomination it takes to make one of the next higher.* See the following example in pounds, shillings and pence.

£	s.	d.
984	12	8
38	6	9
1413	14	10
319	18	2
2756	12	5

First, adding together the pence, because they are the parts of the least value, and taking together both the units and tens of this denomination, we find 29; but as 12 pence make a shilling, this sum amounts to 2 shillings and 5 pence; we then write down only the 5 pence and retain the shillings in order to unite them to the column to which they belong.

Next, we add separately the units and the tens of the next denomination; the first give by joining to them the 2 shillings reserved from the pence, 22; we write down only the two units and retain the two tens for the next column, the sum of which, by this means, amounts to 5 tens, but as the pound, made up of 20 shillings, contains 2 tens, we obtain the number of pounds resulting from the shillings, by dividing the tens of these last by 2; the quotient is 2, and the remainder 1, which last is written under the column to which it belongs, while the pounds are reserved for the next column on the left; as this column is the last, the operation is performed as in simple numbers, and the whole sum is found to be 2756l. 12s. 5d.

The method of proving the addition of compound numbers is derived from the same principles, as that for simple numbers, and is performed in the same manner, care being taken in passing from one denomination to another, to substitute instead of the decimal ratio, the value of each part in the terms of that, which follows it on the right. Let there be, for example,

£	s.	d.
984	12	8
38	6	9
1413	14	10
319	18	2
2756	12	5
1122	22	0

The operation on the pounds is performed according to the rule of article 19; then we change the two pounds into tens of shillings, and obtain 4 of these tens, which joined to that written under the column, makes 5, from which we subtract the 3 units of this column, and place the remainder, 2, underneath, counting it as tens with regard to the next column. There still remain 2 shillings, which must be reduced to pence; adding the result, 24 pence, to the 5 that are written. we have a total of 29, which must be again obtained by the addition of all the pence, as these are the parts of the lowest denomination in the question. This really happens, and proves the operation to be right.

Examples.

	£	s.	d.	£	s.	d.	£	s.	d.
	17	13	4	84	17	$5\frac{1}{2}$	175	10	10
	13	10	2	75	13	$4\frac{1}{4}$	107	13	$11\frac{3}{4}$
	10	17	3	51	17	$8\frac{3}{4}$	89	18	10
	8	8	7	20	10	$10\frac{1}{4}$	75	12	$2\frac{1}{4}$
	3	3	4	17	15	$4\frac{1}{2}$	3	3	$3\frac{3}{4}$
		8	8	10	10	11	1		$\frac{1}{2}$
Sum	54	1	4	261	5	$8\frac{1}{4}$	452	19	$2\frac{1}{4}$
Proof	23	32	0	24	23	20	232	13	0

lb.	oz.	dwt.	gr.
17	3	15	11
13	2	13	13
15	3	14	14
13	10		
12	1		17
		13	14

lb.	oz.	dwt.	gr.
14	10	13	20
13	10	18	21
14	10	10	10
10	1	2	3
1	4	4	4
	1	19	

lb.	oz.	dwt.	gr.
27	10	17	18
17	10	13	13
13	11	13	1
10	1		2
4	4	3	3
2			1

cwt.	qr.	lb.	oz.	dr.
15	2	15	15	15
13	2	17	13	14
12	2	13	14	14
10	1	17	15	
12	1	10		10
10	1	12	1	7

T.	cwt.	qr.	lb.	oz.	dr.
2	17	3	13	8	7
2	13	3	14	8	8
1	16		10		5
2	13			1	7
1	14	1	1	2	2
4	16	1	7	7	5

T.	cwt.	qr.	lb.	oz.	dr.
3	13	2	10	7	7
2	14	1	17	6	6
4	17		14		6
2	13		12	7	7
3	13		10	4	4
5		2	12	8	8

Mls.	fur.	pol	yd	ft.	in.
37	3	14	2	1	5
28	4	17	3	2	10
17	4	4	3	1	2
10	5	6	3	1	7
29	2	2	2		3
30			4		2

Mls	fur	pol.	yd.	ft.	in.
28	2	13	1	1	4
39	1	17	2	2	10
28	1	14	2	2	
48	1	17	2	2	7
37	1	29			3
2		20		2	1

Mls	fur.	pol.	yd.	ft.	in.
28	3	7	2		7
30			1		7
27	6	30	2	2	
7	6	20	2	1	
5	2			2	10
	7	10		2	2

Subtraction of Compound Numbers.

104. THIS operation is performed in the same way as the subtraction of simple numbers, except with regard to the number which it is necessary to borrow from the higher denominations, in order to perform the partial subtractions, when the lower number exceeds the upper. For instance,

	£	s.	d.
from	795	3	0
take	684	17	4
Difference	110	5	8

In performing this example, it is necesary to borrow, from the column of shillings, 1 shilling or 12 pence, in order to effect the subtraction of the lower number, 4, and we have for a remainder 8 pence. There now remain in the upper number of the column of shillings only 2, it is necessary therefore to borrow, from that of pounds, 1 pound or 20 shillings, we thus make it 22, of which, when the lower number, 17, is subtracted, 5 remain; we must now proceed to the column of pounds, remembering to count the upper number less by unity, and finish the operation as in the case of simple numbers.

The method of proving subtraction of compound numbers, like that for simple numbers, consists in adding the difference to the less of the two numbers.

Examples for practice.

	£	s.	d.	£	s.	d.	£	s.	d.
	275	13	4	454	14	$2\frac{3}{4}$	274	14	$2\frac{1}{4}$
	176	16	6	276	17	$5\frac{1}{2}$	85	15	$7\frac{3}{4}$
Rem.	98	16	10	177	16	$9\frac{1}{4}$	188	18	$6\frac{1}{2}$
Proof	275	13	4	454	14	$2\frac{3}{4}$	274	14	$2\frac{1}{4}$

	lb.	oz.	dwt.	gr.	lb.	oz.	dwt.	gr.	lb.	oz.	dwt.	gr.
	7	3	14	11	27	2	10	20	29	3	14	5
	3	7	15	20	20	3	5	21	20	7	15	7
Rem.												
Proof												

	cwt.	qr.	lb.	oz.	dr.	cwt.	qr.	lb.	oz.	dr.	cwt.	qr.	lb.	oz.	dr.
	5		17	5	9	22	2	13	4	8	21	1	7	6	13
	3	3	21	1	7	20	1	17	6	6	13		8	8	14
Rem.															
Proof															

	Mls.	fur.	pol.	yd.	ft.	in.	Mls.	fur.	pol.	yd.	ft.	in.	Mls.	fur.	pol.	yd	ft.	in.
	14	3	17	1	2	1	70	7	13	1	1	2	70	3	10			7
	10	7	30	2		10	20		14	2	2	7	17	3	11	1	1	3
Rem.																		
Proof																		

	m.	w.	d.	h.	′	m.	w.	d.	h.	′	m.	w.	d.	h.	′
	17	2	5	17	26	37	1		13	1	71				5
	10			18	18	15	2		15	14	17		5	5	7
Rem.															
Proof															

Multiplication of Compound Numbers.

105. We have seen, that a number consisting of several denominations may be reduced to a single one, either the lowest or the highest of those contained in it, in which state it admits of being used as an abstract number. But when it is required to find the product of two numbers, one of which only is compound, the simplest method is to consider the multiplication of each denomination of the compound number by the simple factor, as a distinct question, and the several results, thus obtained, will be the total product sought. If it were proposed, for example, to multiply 7l. 14s. 7d. 3q. by 9, it may be done thus,

£	s.	d.	q.
7	14	7	3
9	9	9	9
63	126	63	27

and 63l. 126s. 63d. 27q. is evidently 9 times the proposed sum, because it is 9 times each of the parts, which compose this sum.

But 29q. is equal to 6d. 3q. and adding the 6d. to the 63d. we have 69d. equal to 5s. 9d. adding the 5s. to the 126s. we obtain 131s. equal 6l. 11s, and lastly, adding the 6l. to the 63l. we have 69l. 11s. 9d. 3q. equal to the above result, and equal to the product of

9l. 14s. 7d. 3q. by 9.

Instead of finding the several products first, and then reducing them, we may make the reductions after each multiplication, putting down what remains of this denomination, and carrying forward the quotient, thus obtained, to be united to the next higher product.

Hence, *to multiply two numbers together, one of which is compound, make the compound number the multiplicand and the simple number the multiplier, and beginning with the lowest denomination of the multiplicand, multiply it by the multiplier and divide the product by the number, which it takes to make one of the next superior denomination; putting down the remainder, add the quotient to the product of the next denomination by the multiplier, reduce this sum, putting down the remainder and reserving the quotient, as before, and proceed in this manner through all the denominations to the last, which is to be multiplied like a simple number.*

When the multiplier exceeds 12, that is, when it is so large that it is inconvenient to multiply by the whole at once, the shortest method is to resolve it, if it can be done, into two or more factors, and to multiply first by one and then that product by another, and so on, as in the following example. Let the two numbers be £4 13s. 3d. and 18.

£	s.	d.
4	13	3
		9
41	19	3
		2
83	18	6

Here we first find 9 times the multiplicand, or £41 19s. 3d. and then take twice this product which will evidently be twice 9, or 18 times the original multiplicand (82). Instead of multiplying by 9 we might multiply first by 3 and then that product by 3, which would give the same result; also, the multiplier 18 might be resolved into 3 and 6, which would give the same product as the above. If we multiply £83 18s. 6d. by 7.

£	s.	d.
83	18	6
		7
587	9	6

we shall have the product of the original multiplicand by 7 times 18 or 126.

If the multiplier were 105, it might be resolved into 7, 3 and 5, and the product found as above.

But it frequently happens, that the multiplier cannot be resolved in this way into factors. When this is the case, we may take the number nearest to it, which can be so resolved, and find the product of the multiplicand by this number, as already described, and then add or subtract so many times the multiplicand, as this number falls short, or exceeds the given multiplier, and the result will be the product sought. Let there be £1 7s. 8d. to be multiplied by 17.

	£	s.	d.
	1	7	8
			4
	5	10	8
			4
	22	2	8
	1	7	8
Product	£23	10	4

In the first place, I find the product of £1 7s. 8d. by 16, which is £22 2s. 8d. and to this I add once the multiplicand and this sum £23 10s. 4d. is evidently equal to 17 times the multiplicand.

106. It may be observed, that in those cases, where the decrease of value from one denomination to another, is according to the same law throughout, that is, where it takes the same number of a lower denomination to make one of the next higher through

all the denominations, the multiplication of one compound number by another, may be performed in a manner similar to what takes place with regard to abstract numbers.

This regular gradation is sometimes preserved in the denominations, that succeed to feet in long measure, 1 inch or *prime* being considered as equal to 12 *seconds*, and 1 *second* to 12 *thirds*, and so on, the several denominations after feet being distinguished by one, two, &c. accents, thus,

$$10f.\ 4'\ 5''\ 10'''.$$

If it were required to find the product of 2f. 4′ by 3f. 10′ we should proceed as below.

	f.	′
	2	4
	3	10
1	11	4
7	0	
8	11	4″

The 4 inches or primes may be considered with reference to the denomination of feet, as 4 twelfths, or $\frac{4}{12}$, and the 10 inches as $\frac{10}{12}$, the product of which is $\frac{40}{144}$, or $\frac{40}{12}$ of $\frac{1}{12}$, or 40″, which reduced gives 3′ 4″; putting down the 4″, we reserve the 3′ to be added to the product of 2 feet by 10′, or $\frac{10}{12}$, which product is $\frac{20}{12}$ of a foot, to which 3 being added, we have $\frac{23}{12}$f, or 1f. and 11′; next multiplying 4′ or $\frac{4}{12}$ by 3, we have $\frac{12}{12}$ or 1, which added to the product of 2 by 3, gives 7. Taking the sum of these results, we have 8f. 11′ 4″, for the product of 2f. 4′ by 3f. 10′. The method here pursued may be extended to those cases, where there is a greater number of denominations.

Whence, *to multiply one number consisting of feet, primes, seconds, &c. by another of the same kind, having placed the several terms of the multiplier under the corresponding ones of the multiplicand, multiply the whole multiplicand by the several terms of the multiplier successively according to the rule of the last article, placing the first term of each of the partial products under its respective multiplier, and find the sum of the several columns, observing to carry one for every twelve in each part of the operation; then the* *first number on the left will be feet, and the second primes, and the third seconds, and so on regularly to the last*†.

Examples for practice.

Multiply £1 11s. 6d. 2q. by 5. *Ans.* £7 17s. 8d. 2q.
Multiply 7s. 4d. 3q. by 24. *Ans.* £8 17s. 6d.
Multiply £1 17s. 6d. by 63. *Ans.* £118 2s. 6d.
Multiply 17s. 9d. by 47. *Ans.* £41 14s. 3d.
Multiply £1 2s. 3d. by 117. *Ans.* £130 3s. 3d.

What is the value of 119 yards of cloth at £2 4s. 3d. per yard? *Ans.* £263 5s. 9d.

What is the value of 9cwt. of cheese at £1 11s. 5d. per cwt? *Ans.* £14 2s. 9d.

What is the value of 96 quarters of rye at £1 3s. 4d. per quarter. *Ans.* £112.

What is the weight of 7 hhds. of sugar, each weighing 9cwt. 3grs. 12lb. *Ans.* 69cwt.

In the Lunar circle of 19 years of 365d. 5h. 48′ 48‴ each, how many days, &c.? *Ans.* 6839d. 14h. 27′ 12″.

Multiply 14f. 9′ by 4f. 6′. *Ans.* 66f. 4′ 6″.

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† The above article relates to what is commonly called *duodecimals.* The operation is ordinarily performed by beginning with the highest denomination of the multiplier, and disposing of the several products as in the first example below. The result is evidently the same whichever method is pursued, as may be seen by comparing this example with that of the same question on the right, performed according to the rule in the text. This last arrangement seems to be preferable, as it is more strictly conformable to what takes place in the multiplication of numbers accompanied by decimals.

f.	′	″		
10	4	5		
7	8	1		
72	6	11		
6	10	11	4‴	
	5	2	2	6⁗
79	11	0	6	6

		f.	′	″
		10	4	5
		7	8	6
	5	2	2	6
6	10	11	4	
72	6	11		
79f.	11′	0″	6‴	6⁗

Multiply 4f. 7′ 8″ by 9f. 6′. *Ans.* 44f. 0′ 10″.

Required the content of a floor 48f. 6′ long and 24f. 3′ broad. *Ans.* 1176f. 1′ 6″.

What is the number of square feet, &c. in a marble slab whose length is 5f. 7′ and breadth 1f. 10′? *Ans.* 10f. 2′ 10″.

Division of Compound Numbers.

107. A COMPOUND number may be divided by a simple number, by regarding each of the terms of the former, as forming a distinct dividend. If we take the product found in article 105, namely, £63 126s. 63d. 27q. and divide it by the multiplier 9, we shall evidently come back to the multiplicand, £7 14s. 7d. 3q. We arrive at the same result also, by dividing the above sum reduced, or £69 11s. 9d. 3q. for we obtain one 9th of each of the several parts that compose the number, the sum of which must be one 9th of the whole. But since, in this case, each term of the dividend is not exactly divisible by the divisor, instead of employing a fraction we reduce what remains, and add it to the next lower denomination, and then divide the sum thus formed, by the divisor. The operation may be seen below.

```
£69  11s.  9d.  3q. | 9
 63                 |______________________
 --                 | £7 14s. 7d. 3q.
  6
 20
---
131
  9
---
 41
 36
 --
  5
 12
 --
 69
 63
 --
  6
  4
 --
 27
 27
 --
  0
```

Whence, *to divide a number consisting of different denominations by a simple number, divide the highest term of the compound number by the divisor, reduce the remainder to the next lower denomination, adding to it the number of this denomination, and divide the sum by the divisor, reducing the remainder, as before, and proceed in this way through all the denominations to the last, the remainder of which, if there be one, must have its quotient represented in the form of a fraction by placing the divisor under it. The sum of the several quotients, thus obtained, will be the whole quotient required.*

When the divisor is large and can be resolved into two or more simple factors, we may divide first by one of these factors and then that quotient by another, and so on, and the last quotient will be the same as that which would have been obtained by using the whole divisor in a single operation. Taking the result of the example in the corresponding case of multiplication, we proceed thus,

```
£83  18s.  6d. |    2
 8             |_____________________
 —              £41  19s.  3d. |    9
 3               36            |________________
 2               —              £4  13s.  3d.
 —                5
 1               20
20               ——
 —              119
38               9
2                ——
 —               29
18               27
18               —
 —                2
 0               12
 6               —
 6               27
 —               27
 0               —
                  0
```

By dividing £83 18s. 6d. by 2, we obtain one half of this sum, which being divided by 9, must give one 9th of one half, or one 18th of the whole. The first operation may be considered as separating the dividend into two equal parts, and the second as distributing each of these into nine equal parts, the number of parts therefore will be 18, and being equal, one of them must be one 18th of the whole.

But when the divisor cannot be thus resolved, the operation must be performed by dividing by the whole at once. If the quotient, which we are seeking, were known, by adding it to, or subtracting it from, the dividend a certain number of times, increasing or diminishing the divisor at the same time by as many units, we might change the question into one, whose divisor would admit of being resolved into factors, which would give the same quotient; we should thus preserve the analogy which exists between the multiplication and division of compound numbers. But this cannot be done, as it supposes that to be known, which is the object of the operation.

Multiplication and division, where compound numbers are concerned, mutually prove each other, as in the case of simple numbers. This may be seen by comparing the examples, which are given at length to illustrate these rules.

Examples for practice.

Divide £821 17s. $9\frac{3}{4}$d.	by 4.	*Ans.* £205 9s. $5\frac{1}{4}$d.
Divide £28 2s. $1\frac{1}{2}$d.	by 6.	*Ans.* £4 13s. $8\frac{1}{4}$d.
Divide £57 3s. 7d.	by 35.	*Ans.* £1 12s. 8d.
Divide £23 15s. $7\frac{1}{2}$d.	by 37.	*Ans.* 12s. $10\frac{1}{4}$d.
Divide 1061cwt. 2qrs.	by 28.	*Ans.* 37cwt. 3qrs. 18lb.

Divide 375mls. 2fur. 7pls. 2yds. 1ft. 2in. by 39.

Ans. 9mls. 4fur. 39pls. 2ft. 8in.

If 9 yards of cloth cost £4 3s. $7\frac{1}{2}$d. what is it per yard?

Ans. 9s. 3d. 2q.

If a hogshead of wine cost £33 12s. what is it per gallon?

Ans. 10s. 8d.

If a dozen silver spoons weigh 3lb. 2oz. 13pwt. 12grs. what is the weight of each spoon.

If a person's income be £150 a year, what is it per day?

Ans. 8s. $2\frac{1}{2}$d. nearly.

A capital of £223 16s. 8d. being divided into 96 shares, what is the value of a share? *Ans.* £2 7s. $8\frac{1}{12}$d.

Proportion.

108. We have shown, in the preceding part of this work, the different methods necessary for performing on all numbers, whether whole or fractional, or consisting of different denominations, the four fundamental operations of arithmetic, namely, addition, subtraction, multiplication and division; and all questions relative to numbers ought to be regarded as solved, when, by an attentive examination of the manner in which they are stated, they can be reduced to some one of these operations. Consequently, we might here terminate all that is to be said on arithmetic, for what remains belongs, properly speaking, to the province of algebra. We shall, nevertheless, for the sake of exercising the learner, now resolve some questions which will prepare him for an algebraic analysis, and make him acquainted with a very important theory, that of ratios and proportions, which is ordinarily comprehended in arithmetic.

109. *A piece of cloth* 13 *yards long was sold for* 130 *dollars, what will be the price of a piece of the same cloth* 18 *yards long.*

It is plain, that if we knew the price of one yard of the cloth that was sold, we might repeat this price 18 times, and the result would be the price of the piece 18 yards long. Now, since 13 yards cost 130 dollars, one yard must have cost the thirteenth part of 130 dollars, or $\frac{130}{13}$, performing the division, we find for the result 10 dollars, and multiplying this number by 18, we have 180 dollars for the answer; which is the true cost of the piece 18 yards long.

A courier, who travels always at the same rate, having gone 5 *leagues in* 3 *hours, how many will he go in* 11 *hours?*

Reasoning as in the last example, we see, that the courier goes in one hour $\frac{1}{3}$ of 5 leagues, or $\frac{5}{3}$, and consequently, in 11 hours he will go 11 times as much, or $\frac{5}{3}$ of a league multiplied by 11, or $\frac{55}{3}$, that is 18 leagues and 1 mile.

In how many hours will the courier of the preceding question go 22 *leagues?*

We see, that if we knew the time he would occupy in going one league, we should have only to repeat this number 22 times and the result would be the number of hours required. Now the courier, requiring 3 hours to go 5 leagues, will require only $\frac{1}{5}$ of the time, or $\frac{3}{5}$ of an hour, to go one league; this number, multiplied by 22, gives $\frac{66}{5}$ or 13 hours and $\frac{1}{5}$, that is, 13 hours and 12 minutes.

110. We have discovered the unknown quantities by an analysis of each of the preceding statements, but the known numbers and those required depend upon each other in a manner, that it would be well to examine.

To do this, let us resume the first question, in which it was required to find the price of 18 yards of cloth, of which 13 cost 130 dollars.

It is plain, that the price of this piece would be double, if the number of yards it contained were double that of the first; that, if the number of yards were triple, the price would be triple also, and so on; also that for the half or two thirds of the piece we should have to pay only one half or two thirds of the whole price.

According to what is here said, which all those, who understand the meaning of the terms, will readily admit, we see, that if there be two pieces of the same cloth, the price of the second ought to contain that of the first, as many times as the length of the second contains the length of the first, and this circumstance is stated in saying, that the prices are in proportion to the lengths, or have the same relation to each other as the lengths.

This example will serve to establish the meaning of several terms which frequently occur.

111. The *relation* of the lengths is the number, whether whole or fractional, which denotes how many times one of the lengths contains the other. If the first piece had 4 yards and the second 8, the relation, or ratio, of the former to the latter would be 2, because 8 contains 4 twice. In the above example, the first piece had 13 yards and the second 18, the ratio of the former to the latter is then $\frac{18}{13}$, or $1\frac{5}{13}$. In general, *the relation* or *ratio of two numbers, is the quotient arising from dividing one by the other.*

As the prices have the same relation to each other, that the lengths have, 180 divided by 130 must give $\frac{18}{13}$ for a quotient, which is the case; for in reducing $\frac{180}{130}$ to its most simple terms, we get $\frac{18}{13}$.

The four numbers, 13, 18, 130, 180, written in this order, are then such, that the second contains the first as many times as the fourth contains the third, and thus they form what is called a proportion.

We see also, that *a proportion is the combination of two equal ratios.*

We may observe, in this connexion, that a relation is not changed by multiplying each of its terms by the same number; and this is plain, because a relation, being nothing but the quotient of a division, may always be expressed in a fractional form. Thus the relation $\frac{18}{13}$ is the same as $\frac{180}{130}$.

The same considerations apply also to the second example. The courier, who went 5 leagues in 3 hours, would go twice as far in double that time, three times as far in triple that time; thus 11 hours, the time spent by the courier in going 18 leagues and $\frac{1}{3}$, or $\frac{55}{3}$ of a league, ought to contain 3 hours, the time required in going 5 leagues, as often as $\frac{55}{3}$ contains 5.

The four numbers 5, $\frac{55}{3}$, 3, 11, are then in proportion; and in reality if we divide $\frac{55}{3}$ by 5, we get $\frac{55}{15}$, a result equivalent to $\frac{11}{3}$. It will now be easy to recognise all the cases, where there may be a proportion between the four numbers.

112. To denote that there is a proportion between the numbers 13, 18, 130 and 180, they are written thus,

$$13 : 18 :: 130 : 180,$$

which is read 13 *is to* 18 *as* 130 *is to* 180; that is, 13 is the same part of 18 that 130 is of 180, or that 13 is contained in 18 as many times as 130 is in 180, or lastly, that the relation of 18 to 13 is the same as that of 180 to 130.

The first term of a relation is called the *antecedent*, and the second the *consequent*. In a proportion there are two *antecedents* and two *consequents*, viz. the antecedent of the first relation and that of the second; the consequent of the first relation and that of the second. In the proportion 13 : 18 : 130 : 180, the antecedents are 13, 130; the consequents 18 and 180.

We shall in future take the consequent for the numerator, and the antecedent for the denominator of the fraction which expresses the relation.

113. To ascertain that there is a proportion between the four numbers 13, 18, 130 and 180, we must see if the fractions $\frac{18}{13}$ and $\frac{180}{130}$ be equal, and, to do this, we reduce the second to its most simple terms; but this verification may also be made by considering, that if, as is supposed by the nature of proportion, the two fractions $\frac{18}{13}$ and $\frac{180}{130}$ be equal, it follows that, by reducing them to the same denominator, the numerator of the one will become equal to that of the other, and that, consequently, 18 multiplied by 130 will give the same product as 180 by 13. This is actually the case, and the reasoning by which it is shown, being independent of the particular values of the numbers, proves, that, *if four numbers be in proportion, the product of the first and last, or of the two extremes, is equal to the product of the second and third, or of the two means.*

We see at the same time, that, if the four given numbers were not in proportion, they would not have the abovementioned property; for the fraction, which expresses the first ratio, not being equivalent to that which expresses the second, the numerator of the one will not be equal to that of the other, when they are reduced to a common denominator.

114. The first consequence, naturally drawn from what has been said, is, that the order of the terms of a proportion may be changed, provided they be so placed, that the product of the extremes shall be equal to that of the means. In the proportion 13 : 18 : : 130 : : 180, the following arrangements may be made;

13 : 18 : : 130 : 180
13 : 130 : : 18 : 180
180 : 130 : : 18 : 13
180 : 18 : : 130 : 13
18 : 13 : : 180 : 130
130 : 13 : : 180 : 18
18 : 180 : : 13 : 130
130 : 180 : : 13 : 18;

for in each one of these, the product of the extremes is formed of the same factors, and the product of the means of the same factors. The second arrangement, in which the means have changed places with each other, is one of those that most frequently occur*.

115. This change shows that, we may either multiply or divide the two antecedents, or the two consequents, by the same number, without destroying the proportion. For this change makes the two antecedents to constitute the first relation, and the two consequents, the second. If, for instance, 55 : 21 : : 165 : 63, changing the places of the means we should have,

55 : 165 : : 21 : 63 ;

we might now divide the terms, which form the first relation, by 5, (111) which would give 11 : 33 : : 21 : 63, changing again the places of the means, we should have 11 : 21 : : 33 : 63, a proportion which is true in itself, and which does not differ from the given proportion, except in having had its two antecedents divided by 5.

116. Since the product of the extremes is equal to that of the means, one product may be taken for the other, and, as in dividing the product of the extremes, by one extreme, we must necessarily find the other as the quotient, *consequently, in dividing by one extreme the product of the means, we shall find the other extreme.* For the same reason, *if we divide the product of the extremes by one of the means, we shall find the other mean.*

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* It may be observed, that the proportion 13 : 130 : : 18 : 180 might have been at once presented under this form, according to the solution of the question in article 109; for the value of a yard of cloth may be ascertained in two ways, namely, by dividing the price of the piece of 13 yards by 13, or by dividing the price of 18 yards by 18; it follows then, that the price of the first must contain 13 as many times as the price of the second contains 18; we shall then have, 13 : 130 : : 18 : 180. We may reason in the same manner with respect to the 2nd question in the article above referred to, as well as with respect to all others of the like kind, and thence derive proportions; but the method adopted in article 109 seemed preferable, because it leads us to compare together numbers of the same denomination, whilst by the others we compare prices, which are sums of money, with yards, which are measures of length; and this cannot be done without reducing them both to abstract numbers.

We can then find any one term of a proportion, when we know the other three, for the term sought must be either one of the extremes or one of the means.

The question of article (109) may be resolved by one of these rules. Thus, when we have perceived that the prices of the two pieces are in the proportion of the number of yards contained in each, we write the proportion in this manner,

$$13 : 18 : : 130 : x,$$

putting the letter x instead of the required price of 18 yards, and we find the price, which is one of the extremes, by multiplying together the two means, 18 and 130, which makes 2340, and dividing this product by the known extreme, 13; we obtain, for the result, 180.

The operation, by which, when any three terms of a proportion are given, we find the fourth, is called the *Rule of Three.* Writers on arithmetic have distinguished it into several kinds, but this is unnecessary, when the nature of proportion and the enunciation of the question are well understood; as a few examples will sufficiently show.

117. A person having travelled 217,5 miles in 9 days; it is asked, how long he will be in travelling 423,9 miles, he being supposed to travel at the same rate?

In this question the unknown quantity is the number of days, which ought to contain the 9 days spent in going 217,5 miles, as many times as 423,9 contains 217,5; we thus get the following proportion;

$$217{,}5 : 423{,}9 : : \overset{\text{days}}{9} : x,$$ and we find for x, 17,54 nearly.

118. All the difficulty in these questions, consists in the manner of stating the proportion. The following rules will be sufficient to guide the learner in all cases.

Among the four numbers which constitute a proportion, there are two of the same kind, and two others also of the same kind, but different from the first two. In the preceding examples, two of the terms are miles, and the other two, days.

First, then, it is necessary to distinguish the two terms of each kind, and when this is done, we shall necessarily have the quotient of the greatest term of the second kind by the smallest of the same kind, equal to the quotient of the greatest term of the first kind by the smallest of the same kind, which will give us this proportion,

the smaller term of the first kind
is
to the larger of the same kind
as
the smaller term of the second kind
is
to the larger of this kind.

In the preceding example this rule immediately gives,

$$217{,}5 : 423{,}9 :: 9 : x$$

for the unknown term ought to be greater than 9, since a greater number of days will be necessary to complete a longer journey.

119. If it were required to find how many days it would take 27 men to perform a piece of work, which 15 men, working at the same rate, would do in 18 days; we see that the days should be less in proportion as the number of men is greater, and reciprocally. There is still a proportion in this case, but the order of the terms is inverted; for, if the number of workmen in the second set were triple of that in the first, they would require only one third of the time. The first number of days then would contain the second, as many times as the second number of workmen would contain the first. This order of the terms being the reverse of that assigned to them by the enunciation of the question, we say, that the number of workmen is in the *inverse ratio* of the number of days. If we compare the two first, and the two last, in the order in which they present themselves, the ratio of the former will be 3, or $\frac{3}{1}$, and that of the latter $\frac{1}{3}$, which is the same as the preceding with the terms inverted.

It is evident, indeed, that we invert a ratio by inverting the terms of the fraction, which expresses it, since we make the antecedent take the place of the consequent, and the consequent that of the antecedent. $\frac{3}{2}$ or $2 : 3$ is the inverse of $\frac{2}{3}$, or $3 : 2$.

The mode of proceeding in such cases, may be rendered very simple; for we have only to take the numbers denoting the two sets of workmen, for the quantities of the first kind, and the numbers denoting the days, for those of the second, and to place the one and the other in the order of their magnitude; proceeding thus, we have the following proportion,

$$15 : 27 :: x : 18,$$

from which we immediately find x equal to 10.

Recapitulating the remarks already given. we have the following rule; *make the number which is of the same kind with the answer the third term, and the two remaining ones the first and second, putting the greater or the less first, according as the third is greater or less than the term sought; then the fourth term will be found by multiplying together the second and third, and dividing the product by the first.*

120. 1st. A man placed 3575 dollars at interest at the rate of 5 per cent. yearly; it is asked what will be the interest of this sum at the end of one year?

The expression, 5 per cent. interest, means, that for a sum of one hundred dollars, 5 dollars is allowed at the end of a year; if then, we take the two principals for the quantities of the first kind, and the interest for those of the second, we shall have,

$$100 : 3575 :: 5 : x,$$

a proportion which may be reduced to $20 : 3595 :: 1 : x$, according to the observation in article 115; then dividing the two terms of the first relation, by 5, we shall have $4 : 715 :: 1 : x$, whence x is equal to $\frac{715}{4}$, or $178,75 cts.

We may also resolve this question by considering that 5 is $\frac{1}{20}$ of 100, and that consequently we shall obtain the interest of any sum put out at this rate, by taking the twentieth part of this sum. Now $\frac{1}{20}$ of $3575 is $178,75; a result which agrees with the one before found.

2d. A merchant gives his note for $800,00 payable in a year; the note is sold to a broker, who advances the money for it, eight months before the time of payment; how much ought the broker to give?

As the broker advances from his own funds, a sum, which is not to be replaced till the expiration of 8 months, it is proper that he should be allowed interest for his money during this time.

Let the interest for a year be 6 per cent. the interest for 8 months will be $\frac{8}{12}$, or $\frac{2}{3}$, of 6, or 4; a sum then of 100 dollars lent for 8 months, must be entitled to 4 dollars interest that is, he who borrows it, ought to return \$104. By considering the \$800, as a sum so returned for what is advanced by the broker, we shall have this proportion, 104 : 100 : : 800 : x, whence we get \$769,23† for the value of x, that is, for the sum the broker ought to give.*

Questions for practice.

What is the value of a cwt. of sugar at $5\frac{1}{2}$d. per lb.?

Ans. 2l. 11s. 4d.

What is the value of a chaldron of coals at $11\frac{1}{2}$d. per bushel?

Ans 1l. 14s. 6d.

What is the value of a pipe of wine at $10\frac{1}{2}$d. per pint?

Ans. 44l. 2s.

At 3l. 9s. per cwt. what is the value of a pack of wool, weighing 2cwt. 2qrs. 13lb. *Ans.* 9l. 6d. $\frac{12}{112}$.

What is the value of $1\frac{1}{2}$cwt. of coffee at $5\frac{1}{2}$d. per ounce?

Ans. 61l. 12s.

Bought 3 casks of raisins, each weighing 2cwt. 2qrs. 25lb. what will they come to at 2l. 1s. 8d. per cwt?

Ans. 17l. $4\frac{3}{4}$d. $\frac{32}{112}$.

What is the value of 2qrs. 1nl. of velvet at 19s. $8\frac{1}{2}$d. per English ell? *Ans.* 8s. $10\frac{1}{4}$d. $\frac{14}{20}$.

Bought 12 pockets of hops, each weighing 1cwt. 2qrs. 17lb.; what do they come to at 4l. 1s. 4d. per cwt.?

Ans. 80l. 12s. $1\frac{1}{2}$d. $\frac{96}{112}$.

What is the tax upon 745l. 14s. 8d. at 3s. 6d. in the pound?

Ans. 130l. 10s. $0\frac{3}{4}$d. $\frac{48}{240}$.

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† A sum thus advanced, is called the *present worth* of the sum due at the expiration of the proposed time.

* The operation by which we find what ought to be given for a sum of money, when the time of payment is anticipated, belongs to what is called *Discount*. There are several ways of calculating discount, but the above is the most exact, as it has regard merely to simple interest.

If $\frac{3}{4}$ of a yard of velvet cost 7s. 3d. how many yards can I buy for 13l. 15s. 6d.? *Ans.* 28$\frac{1}{2}$ yards.

If an ingot of gold, weighing 9lb. 9oz. 12dwt. be worth 411l. 12s. what is that per grain? *Ans.* 1$\frac{3}{4}$d.

How many quarters of corn can I buy for 140 dollars at 4s. per bushel? *Ans.* 26qrs. 2bu.

Bought 4 bales of cloth, each containing 6 pieces, and each piece 27 yards, at 16l. 4s. per piece; what is the value of the whole, and the rate per yard?

Ans. 388l. 16s. at 12s. per yard.

If an ounce of silver be worth 5s. 6d. what is the price of a tankard, that weighs 1lb. 10oz. 10dwt. 4gr.?

Ans. 6l. 3s. 9$\frac{1}{2}$d. $\frac{96}{480}$.

What is the half year's rent of 547 acres of land at 15s. 6d. per acre? *Ans.* 211l. 19s. 3d.

At $1,75 per week, how many months board can I have for 100l.? *Ans.* 47m. 2w. $\frac{60}{126}$.

Bought 1000 Flemish ells of cloth for 90l. how must I sell it per ell in Boston to gain 10l. by the whole? *Ans.* 3s. 4d.

If a gentleman's income is 1750 dollars a year, and he spends 19s. 7d. per day, how much will he have saved at the year's end? *Ans.* 167l. 12s. 1d.

What is the value of 172 pigs of lead, each weighing 3cwt. 2qrs. 17$\frac{1}{2}$lb. at 8l. 17s. 6d. per fother of 19$\frac{1}{2}$cwt.?

Ans. 286l. 4s. 4$\frac{1}{2}$d.

The rents of a whole parish amount to 1750l. and a tax is granted of 32l. 16s. 6d. what is that in the pound?

Ans. 4$\frac{1}{2}$d. $\frac{2880}{4200000}$.

If keeping of one horse be 11$\frac{1}{2}$d. per day, what will be that of 11 horses for a year? *Ans.* 192l. 7s. 8$\frac{1}{2}$d.

A person breaking owes in all 1490l. 5s. 10d. and has in money, goods, and recoverable debts, 784l. 17s. 4d. if these things be delivered to his creditors, what will they get on the pound? *Ans.* 10s. 6$\frac{1}{4}$d. $\frac{20993}{35767}$.

What must 40s. pay towards a tax, when 652l. 13s. 4d. is assessed at 83l. 12s. 4d.? *Ans.* 5s. 1$\frac{1}{4}$d. $\frac{15376}{15664}$.

Bought 3 tuns of oil for 151l. 14s. 85 gallons of which being damaged, I desire to know how I may sell the remainder per gallon, so as neither to gain nor lose by the bargain?

Ans. 4s. $6\frac{1}{4}$d. $\frac{25}{671}$.

What quantity of water must I add to a pipe of mountain wine, valued at 33l. to reduce the first cost to 4s. 6d. per gallon?

Ans. $20\frac{2}{3}$ gallons.

If 15 ells of stuff, $\frac{3}{4}$ yard wide, cost 37s. 6d. what will 40 ells of the same stuff cost, being yard wide? *Ans.* 6l. 13s. 4d.

Shipped for Barbadoes 500 pairs of stockings at 3s. 6d. per pair, and 1650 yards of baize at 1s. 3d. per yard, and have received in return 348 gallons of rum at 6s. 8d. per gallon, and 750lb. of indigo at 1s. 4d. per lb. what remains due upon my adventure? *Ans.* 24l. 12s. 6d.

If 100 workmen can finish a piece of work in 12 days, how many are sufficient to do the same in 3 days? *Ans.* 400 men.

How many yards of matting, 2ft. 6in. broad, will cover a floor, that is 27ft. long, and 20ft. broad? *Ans.* 72 yards.

How many yards of cloth, 3qrs. wide, are equal in measure to 30 yards 5qrs. wide? *Ans.* 50 yards.

A borrowed of his friend B 250l. for 7 months, promising to do him the like kindness; sometime after B had occasion for 300l. how long may he keep it to receive full amends for the favor? *Ans.* 5 months and 25 days.

If, when the price of a bushel of wheat is 6s. 3d. the penny loaf weigh 9oz. what ought it to weigh when wheat is at 8s. $2\frac{1}{2}$d. per bushel? *Ans.* 6oz. 13dr.

If $4\frac{1}{2}$cwt. can be carried 36 miles for 35 shillings, how many pounds can be carried 20 miles for the same money?

Ans. 907lb. $\frac{4}{20}$.

How many yards of canvass, that is ell wide, will line 20 yards of say, that is 3qrs. wide? *Ans.* 12yds.

If 30 men can perform a piece of work in 11 days, how many men will accomplish another piece of work, 4 times as big, in a fifth part of the time? *Ans.* 600.

A wall that is to be built to the height of 27 feet, was raised 9 feet by 12 men in 6 days; how many men must be employed to finish the wall in 4 days at the same rate of working?

Ans. 36.

If $\frac{5}{7}$oz. cost $\frac{11}{12}$l. what will 1oz cost? *Ans.* 1l. 5s. 8d.

If $\frac{3}{16}$ of a ship cost 273l. 2s. 6d. what is $\frac{5}{32}$ of her worth? *Ans.* 227l. 12s. 1d.

At $1\frac{1}{2}$l. per cwt. what does $3\frac{1}{3}$lb. come to? *Ans.* $10\frac{5}{7}$d.

If $\frac{5}{8}$ of a gallon cost $\frac{5}{8}$l. what will $\frac{5}{9}$ of tun cost? *Ans.* 140l.

A person, having $\frac{3}{5}$ of a coal mine, sells $\frac{3}{4}$ of his share for 171l. what is the whole mine worth? *Ans.* 380l.

If, when the days are $13\frac{5}{8}$ hours long, a traveller perform his journey in $35\frac{1}{2}$ days; in how many days will he perform the same journey, when the days are $11\frac{9}{10}$ hours long? *Ans.* $40\frac{615}{952}$ days.

A regiment of soldiers, consisting of 976 men, are to be new clothed, each coat to contain $2\frac{1}{2}$ yards of cloth, that is $1\frac{5}{8}$yd. wide, and to be lined with shalloon, $\frac{7}{8}$yd. wide; how many yards of shalloon will line them? *Ans.* 4531yds. 1qr. $2\frac{6}{7}$nl.

Compound Proportion.

121. PROPORTION is also applied to questions, in which the relation of the quantity required, to the given quantity of the same kind, depends upon several circumstances, combined together; it is then called *Compound Proportion,* or *Double Rule of Three.* See some examples.

It is required to find how many leagues a person would go in 17 days, travelling 10 hours a day, when he is known to have travelled 112 leagues, in 29 days, employing only 7 hours a day.

This question may be resolved in two ways, we will first give the one that leads to Compound Proportion.

In each case, the number of leagues passed over depends upon two circumstances, namely, the number of days the man travels, and the number of hours he travels in each day.

We will not at first consider this latter circumstance, but suppose the number of hours to be the same in each case; the question then will be; *a person in* 29 *days, travels* 112 *leagues, how many will he travel in* 17 *days?* This will furnish the following proportion:

$$29 : 17 :: 112 : x.$$

The fourth term will be equal to 112 multiplied by 17 and divided by 29, or $\frac{1904}{29}$ leagues.

Now, to take into consideration the number of hours, we must say, if a person travelling 7 hours a day, for a certain number of days, has travelled $\frac{1904}{29}$ leagues, how far will he go in the same time, if he travel 10 hours a day? This will lead to the following proportion,

$$\overset{h.}{7} : \overset{h.}{10} : \overset{l.}{\tfrac{1904}{29}} : x,$$

which gives for the fourth term, or answer, 93,793 leagues nearly.

The question may also be resolved by observing, that 29 days travelling, at 7 hours a day, is equal to 203 hours travelling; and that 17 days, at 10 hours a day, amounts to 170 hours; the problem then is reduced to this proportion,

$$203 : 170 :: 112 : x,$$

by which we find the distance he ought to travel in 170 hours, according to what he performed in 203 hours.

We see, by the first mode of resolving the question, that 112 leagues has to the fourth term, or answer, the same proportion, that 29 days has to 17, and that 7 hours has to 10, stating this in the form of a proportion, we have

$$\left.\begin{matrix} \overset{d.}{29} : \overset{d.}{17} \\ \overset{h.}{7} : \overset{h.}{10} \end{matrix}\right\} :: \overset{lea.}{112} : x,$$

by which it appears, that 112 is to be multiplied by both 17 and 10, and to be divided by both 29 and 7, that is 112 is to be multiplied by the product of 17 by 10, and divided by the product of 29 by 7, which is the same as the second method of resolving the question.

122. Again, if 9 labourers, working 8 hours a day, have spent 24 days in digging a ditch 65 yards long, 7 wide, and 5 deep, how many days will it take 71 labourers of equal ability, working 11 hours a day, to dig a ditch 327 yards long, 18 broad and 7 deep?

Here is a question very complicated in appearance, but which may be resolved by proportion.

If all the conditions of these two cases were alike, except the number of men and the number of days, the question would consist only in finding in how many days 71 men would perform the work, which 9 men have done in 24 days; we should have then,

$$71 : 9 :: 24 : x,$$

but instead of calculating the number of days, we will only indicate, as in article 82, the numbers to be multiplied together, and place as a denominator those by which they are to be divided; we thus have for x days,

$$\frac{24 \text{ by } 9}{71}.$$

But as the first labourers worked only 8 hours a day while the others worked 11, the number of days required by the latter will be less in proportion, which will give

$$11 : 8 :: \frac{24 \text{ by } 9}{21} : x;$$

whence we conclude that the number of days, in this case, is

$$\frac{24 \text{ by } 9 \text{ by } 8}{71 \text{ by } 11}.$$

This number is that of the days necessary for 71 labourers, working 11 hours a day, to dig the first ditch.

The ditches being of unequal length, as many more days will be necessary, as the second is longer than the first, thus we shall have

$$65 : 327 :: \frac{24 \text{ by } 9 \text{ by } 8}{71 \text{ by } 11} : x,$$

and the number of days, this new circumstance being considered, will be

$$\frac{24 \text{ by } 9 \text{ by } 8 \text{ by } 327}{71 \text{ by } 11 \text{ by } 65}.$$

Taking into consideration the breadths, which are not alike, we have

$$13 : 18 :: \frac{24 \text{ by } 9 \text{ by } 8 \text{ by } 327}{71 \text{ by } 11 \text{ by } 65} : x,$$

and, consequently, the number of days required changes to

$$\frac{24 \text{ by } 9 \text{ by } 8 \text{ by } 327 \text{ by } 18}{71 \text{ by } 11 \text{ by } 65 \text{ by } 13}.$$

Lastly, the depths being different, we have

$$5 : 7 :: \frac{24 \text{ by } 9 \text{ by } 8 \text{ by } 327 \text{ by } 18}{71 \text{ by } 11 \text{ by } 65 \text{ by } 13} : x,$$

and the number of days, resulting from the concurrence of all the circumstances, is

$$\frac{24 \text{ by } 9 \text{ by } 8 \text{ by } 327 \text{ by } 18 \text{ by } 7}{71 \text{ by } 11 \text{ by } 65 \text{ by } 13 \text{ by } 5}.$$

Performing the multiplications and divisions, we get the answer required, 21 days $\frac{1902831}{3299725}$.

123. This number is equal to 24 multiplied by the fractional quantity

$$\frac{9 \text{ by } 8 \text{ by } 327 \text{ by } 18 \text{ by } 7}{71 \text{ by } 11 \text{ by } 65 \text{ by } 13 \text{ by } 5};$$

but this last quantity, which expresses the relation of the number of days given, to the number of days required, is itself the product of the following fractions;

$$\tfrac{9}{71}, \tfrac{8}{11}, \tfrac{327}{65}, \tfrac{18}{13}, \tfrac{7}{5}.$$

Now, going back to the denominations given to these numbers in the statement of the question, we see that these fractions are the ratios of the men and the hours, of the lengths, the breadths and the depths, of the two ditches; hence it follows, that the ratio of the number of days given, to the number of days sought, is equal to the product of all the ratios, which result from a comparison of the terms relating to each circumstance of the question.

This may be resolved in a simple manner by first finding the value of each of these ratios; for, by multiplying together the fractions that express them, we form a fraction which represents the ratio of the quantity required to the given quantity of the same kind.

This fraction, which will be the product of all the ratios in the question, will have for its numerator the product of all the antecedents, and for its denominator, that of all the consequents. A ratio resulting, in this manner, from the multiplication of several others, is called a *compound ratio*.

By means of the fractional expression

$$\frac{9 \text{ by } 8 \text{ by } 327 \text{ by } 18 \text{ by } 7}{71 \text{ by } 11 \text{ by } 65 \text{ by } 13 \text{ by } 5},$$

and the given number of days, 24, we make the following proportion, 71 by 11 by 65 by 13 by 5 : 9 by 8 by 327 by 18 by 7 : : 24 : x, which may also be represented in this manner, as was the preceding example.

$$\left.\begin{array}{r} 71 : 9 \\ 11 : 8 \\ 65 : 327 \\ 13 : 18 \\ 5 : 7 \end{array}\right\} : : 24 : x.$$

Our remarks may be summed up in this rule ; *Make the number which is of the same kind with the required answer, the third term ; and of the remaining numbers, take any two that are of the same kind and place one for a first term and the other for a second term according to the directions in simple proportion ; then any other two of the same kind, and so on, till all are used ; lastly, multiply the third term by the product of the second terms, and divide the result by the product of the first terms, and the quotient will be the fourth term, or answer required.*

Examples for practice.

If 100l. in one year gain 5l. interest, what will be the interest of 750l. for 7 years? *Ans.* 262l. 10s.

What principal will gain 262l. 10s. in 7 years, at 5l. per cent. per annum? *Ans.* 750l.

If a footman travel 130 miles in 3 days, when the days are 12 hours long ; in how many days, of 10 hours each, may he travel 360 miles? *Ans.* $9\frac{63}{65}$ days.

If 120 bushels of corn can serve 14 horses 56 days ; how many days will 94 bushels serve 6 horses? *Ans.* $102\frac{16}{45}$ days.

If 7oz. 5dwt. of bread be bought at $4\frac{3}{4}$d. when corn is at 4s. 2d. per bushel, what weight of it may be bought for 1s. 2d. when the price per bushel is 5s. 6d.? *Ans.* 1lb. 4oz. $3\frac{479}{627}$ dwt.

If the transporation of 13cwt. 1qr. 72 miles be 2l. 10s. 6d. what will be the transportation of 7cwt. 3qrs. 112 miles?

Ans. 2l. 5s. 11d. $1\frac{77}{139}$q.

A wall, to be built to the height of 27 feet, was raised to the height of 9 feet by 12 men in 6 days ; how many men must be employed to finish the wall in 4 days, at the same rate of working? *Ans.* 36 men.

If a regiment of soldiers, consisting of 939 men consume 351 quarters of wheat in 7 months; how many soldiers will consume 1464 quarters in 5 months, at that rate? *Ans.* $5483\frac{23}{195}$.

If 248 men, in 5 days of 11 hours each, dig a trench 230 yards long, 3 wide and 2 deep; in how many days of 9 hours long, will 24 men dig a trench of 420 yards long, 5 wide and 3 deep? *Ans.* $288\frac{89}{207}$.

Fellowship.

124. THE object of this rule is to divide a number into parts, which shall have a given relation to each other; we shall see in the following example its origin, and whence it has its name.

Three merchants formed a company for the purpose of trade; the first advanced 25000 dollars, the second 18000, and the third 42000; after some time they separated, and wished to divide the joint profit, which amounted to 57225 dollars; how much ought each one to have?

To resolve this question we must consider, that each man's gain ought to have the same relation to the whole gain, as the money he advanced has to the whole sum advanced; for he, who furnishes a half or third of this sum, ought, plainly, to have a half or third of the profit. In the present example, the whole sum being 85000 dollars, the particular sums will be respectively $\frac{25000}{85000}$, $\frac{18000}{85000}$, $\frac{42000}{85000}$ of it; and if we multiply these fractions by the whole gain, 57225, we shall obtain the gain belonging to each man. It is moreover evident, that the sum of the parts will be equal to the whole gain, because the sum of the above fractions, having its numerator equal to its denominator, is necessarily an unit.

We have therefore, these proportions;

$ $ $

85000 : 25000 : : 57225 : to the first man's gain,
85000 : 18000 : : 57225 : to the second man's gain,
85000 : 42000 : : 57225 : to the third man's gain,

which may be enunciated thus;

The whole sum advanced : to each man's particular sum : : the whole gain : to each man's particular gain.

By simplifying the first ratio of each of these proportions we have

$

85 : 25 : : 57225 : to the gain of the 1st. or $$16830\frac{75}{85}$,

85 : 18 : : 57225 : to the gain of the 2d. or $$12118\frac{21}{85}$,

85 : 42 : : 57225 : to the gain of the 3d. or $$28275\frac{75}{85}$.

If all the sums advanced had been equal, the operation would have been reduced to dividing the whole gain by the number of sums advanced; we may reduce the question to this in the present case, by supposing the whole sum, $85000, divided into 85 partial sums, or stocks of $1000 each, the gain of each of these sums will evidently be the 85th. part of the whole gain; and nothing remains to be done, except to multiply this part severally by 25, 18, and 42, considering the sums 25000, 18000 and 42000 as the amounts of 25 shares, 18 shares and 42 shares.

In commercial language the money advanced is called the *capital* or *stock*, and the gain to be divided, the *dividend*.

The following question is very similar to that just resolved.

125. It is required to divide an estate of 67250 dollars among 3 heirs, in such a manner, that the share of the second shall be $\frac{2}{5}$ of that of the first, and the share of the third $\frac{7}{8}$ of that of the second.

It is plain that the share of the third, compared with that of the first, will be $\frac{7}{8}$ of $\frac{2}{5}$ of it, or $\frac{7}{20}$; then the three parts required will be to each other in the proportion of the numbers 1, $\frac{2}{5}$ and $\frac{7}{20}$. Reducing these to a common denominator, we find them $\frac{20}{20}$, $\frac{8}{20}$, and $\frac{7}{20}$, and have the three numbers 20, 8 and 7, which are proportional to the first; but as their sum is 35, it is plain, that if we take three parts expressed by the fractions, $\frac{20}{35}$, $\frac{8}{35}$, and $\frac{7}{35}$, they will be in the required proportion. The question will then be resolved by taking $\frac{20}{35}$, then $\frac{8}{35}$ and then $\frac{7}{35}$ of 67250 dollars, which will give the sums due to the heirs, according to the manner prescribed, namely;

$$38428\frac{20}{35}$, $$15371\frac{15}{35}$, and $13450.

126. Again, there are two fountains, the first of which will fill a certain reservoir in $2\frac{1}{2}$ hours, and the second will fill the same reservoir in $3\frac{3}{4}$ hours; how much time will be required to

fill the reservoir, by means of both fountains running at the same time?

We must first ascertain what part of the reservoir will be filled by the first fountain in any given time, an hour for instance. It is evident that, if we take the contents of the reservoir for unity, we have only to divide 1 by $2\frac{1}{2}$, or $\frac{5}{2}$, which gives us $\frac{2}{5}$ for the part filled in one hour by the first fountain. In the same manner, dividing 1 by $3\frac{3}{4}$, or $\frac{15}{4}$, we obtain $\frac{4}{15}$ for the part of the reservoir filled in an hour by the second fountain; consequently, the two fountains, flowing together for an hour, will fill $\frac{2}{5}$ added to $\frac{4}{15}$, or $\frac{10}{15}$ of the reservoir. If we now divide 1, or the contents of the reservoir, by $\frac{10}{15}$, we shall obtain the number of hours necessary for filling it at this rate; and shall find it to be $\frac{15}{10}$, or an hour and a half.

Authors who have written upon arithmetic, have multiplied and varied these questions in many ways, and have reduced to rules the processes which serve to resolve them; but this multiplication of precepts, is, at least, useless, because a question of the kind we have been considering, may always be solved with facility by one who perceives what follows from the enunciation; especially when he can avail himself of the aid of algebra; we shall therefore proceed to another subject.

Besides the proportions composed of four numbers, one of the two first of which contains the other as many times as the corresponding one of the two last, contains the other; it has been usual to consider as such the assemblage of four numbers, such as 2, 7, 9, 14, of which one of the two first exceeds the other as much as the corresponding one of the two last, exceeds the other.

These numbers, which may be called *equidifferent*, possess a remarkable property, analogous to that of proportion, for the sum of the extreme terms, 2 and 14, is equal to the sum of the means, 7 and 9*.

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* The ancients kept the theory of proportions very distinct from the operations of arithmetic. Euclid gives this theory in the fifth book of his elements, and as he applies the proportions to lines, it is apparent, that we thence derive the name of *geometrical proportion;*

To show this property to be general, we must observe, that the second term is equal to the first increased by the difference, and that the fourth is equal to the third increased by the difference; hence it follows, that the sum of the extremes, composed of the first and fourth terms, must be equal to the first increased by the third increased by the difference. Also, that the sum of the means, composed of the second and third terms, must be equal to the first increased by the difference increased by the third term; these two sums, being composed of the same parts, must consequently be equal.

We have gone on the supposition, that the second and fourth terms were greater than the first and third; but the contrary may be the case, as in the four numbers 8, 5, 15, 12; the second term will be equal to the first decreased by the difference, and the fourth will be equal to the third decreased by the difference. By changing the word *increased* into *decreased*, in the preceding reasoning, it will be proved that, in the present case, the sum of the extremes is equal to that of the means.

We shall not pursue this theory of equidifferent numbers further, because, at present, it can be no use.

Questions for practice.

A and B have gained by trading $182. A put into stock $300 and B $400; what is each person's share of the profit?

Ans. A $78 and B $104.

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and that the name of *arithmetical proportion* was given to an assemblage of equidifferent numbers, which were not treated of till a much later period. These names are very exceptionable; the word *proportion* has a determinate meaning, which is not at all applicable to equidifferent numbers. Besides, the proportion called *geometrical*, is not less arithmetical than that which exclusively possesses the latter name. M. Lagrange, in his Lectures at the Normal school, has proposed a more correct phraseology, and I have thought proper to follow his example.

Equidifference, or the assemblage of four equidifferent numbers, or arithmetical proportion, is written thus; 2 . 7 : 9 . 14.

Among English writes the following form is used;

2 . . 7 : : 9 . . 14.

Divide $120 between three persons, so that their shares shall be to each other as 1, 2, and 3, respectively.

Ans. $20, $40, and $60.

Three persons make a joint stock. A put in $185,66, B $98,50, and C $76,85; they trade and gain $222; what is each person's share of the gain?

Ans. A $104,17$\frac{83}{36101}$, B $60,57$\frac{6243}{36101}$, and C 47,25$\frac{29775}{36101}$.

Three merchants A, B, and C, freight a ship with 340 tuns of wine; A loaded 110 tuns, B 97, and C the rest. In a storm the seamen were obliged to throw 85 tuns overboard; how much must each sustain of the loss? *Ans.* A 27$\frac{1}{2}$, B 24$\frac{1}{4}$, and C 33$\frac{1}{4}$.

A ship worth $860 being entirely lost, of which $\frac{1}{8}$ belonged to A, $\frac{1}{4}$ to B, and the rest to C; what loss will each sustain, supposing $500 of her to be insured? *Ans.* A $45, B $90, and C $225.

A bankrupt is indebted to A $277,33, to B $305,17, to C $152, and to D $105. His estate is worth only $677,50; how must it be divided? *Ans.* A$223,81$\frac{2580}{8395}$, B $246,28$\frac{6115}{8395}$,
C $122,66$\frac{6930}{8395}$, and D $84,73$\frac{6665}{8395}$.

A and B, venturing equal sums of money, clear by joint trade $154. By agreement A was to have 8 per cent. because he spent his time in the execution of the project, and B was to have only 5 per cent.; what was A allowed for his trouble?

Ans. $35,53$\frac{11}{13}$.

Three graziers hired a piece of land for $60,50. A put in 5 sheep for 4$\frac{1}{2}$ months, B put in 8 for 5 months, and C put in 9 for 6$\frac{1}{2}$ months; how much must each pay of the rent?

Ans. A $11,25, B $20, and C $29,25.

Two merchants enter into partnership for 18 months; A put into stock at first $200, and at the end of 8 months he put in $100 more; B put in at first $550, and at the end of 4 months took out $140. Now at the expiration of the time they find they have gained $526; what is each man's just share?

Ans. A's $192,95$\frac{70}{1254}$.
B's 333,04$\frac{1184}{1254}$.

A, with a capital of $1000, began trade January 1, 1776, and meeting with success in business he took in B a partner, with a capital of $1500 on the first of March following. Three months after that, they admit C as a third partner, who brought into stock $2800, and after trading together till the first of the next year, they find the gain, since A commenced business, to be $1776,50. How must this be divided among the partners?

Ans. A's $457,46$\frac{364}{466}$.
B's 571,83$\frac{222}{466}$.
C's 747,19$\frac{346}{466}$.

Alligation.

128. We shall not omit the rule of alligation, the object of which is to find the mean value of several things of the same kind, of different values; the following examples will sufficiently illustrate it.

A wine merchant bought several kinds of wine, namely;

130 bottles which cost him 10 cents each,	
75	at 15
231	at 12
27	at 20;

he afterwards mixed them together; it is required to ascertain the cost of a bottle of the mixture. It will be easily perceived, that we have only to find the whole cost of the mixture, and the number of bottles it contains, and then to divide the first sum by the second, to obtain the price required.

Now, the	130 bottles	at 10 cents	cost	1300 cents
	75	at 15	cost	1125,
	231	at 12	cost	2772,
	27	at 20	cost	540,
then	463 bottles cost			5737 cents.

5737 divided by 463 give 12,39 cents, the price of a bottle of the mixture.

The preceding rule is also used for finding a mean of different results, given by experiment or observation, which do not agree with each other. If, for instance, it were required to know the distance between two points considerably removed from each other, and it had been measured; whatever care might have been used in doing this, there would always be a

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little uncertainty in the result, on account of the errors inevitably committed by the manner of placing the measures one after the other.

We will suppose that the operation has been repeated several times, in order to obtain the distance exactly, and that twice it has been found 3794yds. 2ft. that three other measurements have given 3795yds. 1ft. and a third 3793yds. As these numbers are not alike, it is evident that some must be wrong and perhaps all. To obtain the means of diminishing the error, we reason thus; if the true distance had been obtained by each measurement, the sum of the results would be equal to six times that distance, and it is plain that this would also be the case, if some of the results obtained were too little, and others too great, so that the increase, produced by the addition of the excesses, should make up for what the less measurements wanted of the true value. We should then, in this last case, obtain the true value by dividing the sum of the results by the number of them.

This case is too peculiar to occur frequently, but it almost always happens, that the errors on one side destroy a part of those on the other, and the remainder, being equally divided among the results, becomes smaller according as the number of results is greater.

According to these considerations we shall proceed, as follows;

	yds.	ft.			ft.
We take twice	3794	2	or	7589	1
	yds.	ft.			
3 times	3795	1	or	11386	0
	yds.				
once	3793		or	3793	
6 results, giving in all				22768	1.

Dividing 22768yds. 1ft. by 6, we find the mean value of the required distance to be 3794yds. 2ft.

129. Questions sometimes occur, which are to be solved by a method, the reverse of that above given. It may be required, for example, to find what quantity of two different ingredients it would take to make a mixture of a certain value. It is evident, that if the value of the mixture required exceeds that of one of the ingredients just as much as it falls short of that of the other, we should take equal quantities of each to make the compound.

So also, if the value of the mixture exceeded that of one twice as much as it fell short of that of the other, the proportion of the ingredients would be as one half to one. To mix wine at \$2 per gallon with wine at \$3, so that the compound shall be worth \$2,50, we should take equal quantities, or quantities in the proportion of 1 to 1. If the mixture were required to be worth \$2,66⅔, the quantities would be in the proportion of ½ to 1, or of $\frac{1}{66\frac{2}{3}}$ to $\frac{1}{33\frac{1}{3}}$; and generally, the nearer the mixture rate is to that of one of the ingredients, the greater must be the quantity of this ingredient with respect to the other, and the reverse; hence, *To find the proportion of two ingredients of a given value, necessary to constitute a compound of a required value, make the difference between the value of each ingredient and that of the compound the denominator of a fraction, whose numerator is one, and these fractions will express the proportion required;* and being reduced to a common denominator, the numerators will express the same proportion, or show what quantity of each ingredient is to be taken to make the required compound.

When the compound is limited to a certain quantity, the proportion of the ingredients, corresponding to it, may be found by saying; as the whole quantity, found us above, is to the quantity required, so is each part, as obtained by the rule, to the required quantity of each.

Let it be required, for example, to mix wine at 5s. per gallon and 8s. per gallon, in such quantities that there may be 60 gallons worth 6s. per gallon. The difference between 6s. and 5s. is 1, and between 6s. and 8s. is 2, giving for the required quantities the ratio of $\frac{1}{1}$ to $\frac{1}{2}$, or 2 to 1; thus, taking x equal to the quantity at 5s. and y equal to the quantity at 8s. we have these proportions; $3 : 60 :: 2 : x$, and $3 : 60 :: 1 : y$, giving, for the answer, 40 gallons at 5s. and 20 gallons at 8s. per gallon.

Also, when one of the ingredients is limited, we may say; as the quantity of the ingredient found as above, is to the required quantity of the same, so is the quantity of the other ingredient to the proportional part required.

For example, I would know how many gallons of water at 0s. per gallon, I must mix with thirty gallons of wine at 6s. per gallon, so that the compound may be worth 5s. per gallon. First, the difference between 0s. and 5s. is 5; and the difference between 6s. and 5s. is 1: the quantity of water therefore will be to that of the wine, as $\frac{1}{5}$ to $\frac{1}{1}$, or as 1 to 5. Then, from this ratio, we institute the proportion, $5 : 30 :: 1 : x$, which gives 6, for the number of gallons required.

As we have found the proportion of two ingredients necessary to form a compound of a required value, so also we may consider either of these in connexion with a third, with a fourth, and so on, thus making a compound of any required value, consisting of any number whatever of simple ingredients. The two ingredients used, however, must always be, one of a greater and the other of a less value, than that of the compound required.

A grocer would mix teas at 12s. and 10s. with 40lbs. at 4s. per pound, in such proportions that the composition shall be worth 8s. per lb. If he mix only two kinds, the one at 4s. and the one at 10s. their quantities will be in the ratio of $\frac{1}{4}$ to $\frac{1}{2}$, or 1 : 2; and if he mix the tea at 4s. also with that at 12s. their ratio will be that of $\frac{1}{4}$ to $\frac{1}{4}$, or of 1 to 1. Adding together the proportions of the ingredient, which is taken with each of the others, we find the several quantities, at 4s. 10s. and 12s. to be as 2, 2, and 1. And taking x for the number of lbs. at 10s. and y for the quantity at 12s. we have the following proportions;

$$2 : 40 :: 2 : x; \text{ and } 2 : 40 :: 1 : y;$$

giving, for the answer, 40lb. at 10s and 20lb. at 12s. per pound.

The problems of the two last articles are generally distinguished by the names of alligation *medial*, and alligation *alternate*. A full explanation of the latter belongs properly to algebra.

Examples.

A composition being made of 5lb. of tea at 7s. per pound, 9lb. at 8s. 6d. per pound, and 14½lb. at 5s. 10d. per pound; what is a pound of it worth? *Ans.* 6s. 10½d.

How much gold of 15, of 17, and of 22 carats† fine must be mixed with 5oz. of 18 carats fine, so that the composition may be 20 carats fine? *Ans.* 5oz. of 15 carats fine, 5 of 17, and 25 of 22.

† A carat is a twenty fourth part, 22 carats fine means $\frac{22}{24}$ of pure metal. A carat is also divided into four parts, called grains of a carat.

Miscellaneous Questions for practice.

What number, added to the thirty-first part of 3813, will make the sum 200? *Ans.* 77.

The remainder of a division is 325, the quotient 467, and the divisor is 43 more than the sum of both; what is the dividend? *Ans.* 390270.

Two persons depart from the same place at the same time; the one travels 30, the other 35 miles a day; how far are they distant at the end of 7 days, if they travel both the same road; and how far, if they travel in contrary directions? *Ans.* 35, and 455 miles.

A tradesman increased his estate annually by 100l. more than $\frac{1}{4}$ part of it, and at the end of 4 years found, that his estate amounted to 10342l. 3s. 9d. What had he at first? *Ans.* 4000l.

Divide 1200 acres of land among A, B, and C, so that B may have 100 more than A, and C 64 more than B. *Ans.* A 312, B 412, and C 476.

Divide 1000 crowns; give A 120 more, and B 95 less, than C. *Ans.* A 445, B 230, C 325.

What sum of money will amount to 132l. 16s. 3d. in 15 months, at 5 per cent. per annum, simple interest? *Ans.* 125l.

A father divided his fortune among his sons, giving A 4 as often as B 3, and C 5 as often as B 6; what was the whole legacy, supposing A's share 5000l.? *Ans.* 11875l.

If 1000 men, besieged in a town with provisions for 5 weeks, each man being allowed 16oz. a day, were reinforced with 500 men more. On hearing, that they cannot be relieved till the end of 8 weeks, how many ounces a day must each man have, that the provision may last that time? *Ans.* $6\frac{2}{3}$oz.

What number is that, to which if $\frac{2}{7}$ of $\frac{5}{9}$ be added, the sum will be 1? *Ans.* $\frac{53}{63}$.

A father dying left his son a fortune, $\frac{1}{4}$ of which he spent in 8 months; $\frac{3}{7}$ of the remainder lasted him twelve months longer; after which he had only 410l. left. What did his father bequeath him? *Ans.* 956l. 13s. 4d.

A guardian paid his ward 3500l. for 2500l. which he had in his hands 8 year. What rate of interest did he allow him?

Ans. 5 per cent.

A person, being asked the hour of the day, said, the time past noon is equal to $\frac{4}{5}$ of the time till midnight. What was the time?

Ans. 20min. past 5.

A person, looking on his watch, was asked, what was the time of the day; he answered, it is between 4 and 5; but a more particular answer being required, he said, that the hour and minute hands were then exactly together. What was the time?

Ans. $21\frac{9}{11}$ minutes past 4.

With 12 gallons of Canary, at 6s. 4d. a gallon, I mixed 18 gallons of white wine, at 4s. 10d. a gallon and 12 gallons of cider, at 6s. 1d. a gallon. At what rate must I sell a quart of this composition, so as to clear 10 per cent.? *Ans.* 1s. $3\frac{5}{7}$d.

What length must be cut off a board, $8\frac{3}{8}$ inches broad, to contain a square foot; or as much as 12 inches in length and 12 in breadth? *Ans.* $17\frac{13}{67}$in.

What difference is there between the interest of 350l. at 4 per cent. for 8 years, and the discount of the same sum, at the same rate, and for the same time? *Ans.* 27l. $3\frac{1}{33}$s.

A father devised $\frac{7}{18}$ of his estate to one of his sons, and $\frac{7}{18}$ of the residue to another, and the surplus to his relict for life; the children's legacies were found to be 257l. 3s. 4d. different. What money did he leave for the widow? *Ans.* 635l. $10\frac{30}{49}$d.

What number is that, from which if you take $\frac{3}{7}$ of $\frac{3}{8}$, and to the remainder add $\frac{7}{16}$ of $\frac{1}{26}$, the sum will be 10? *Ans.* $10\frac{191}{2240}$.

A man dying left his wife in expectation, that a child would be afterward added to the surviving family; and making his will ordered, that if the child were a son, $\frac{2}{3}$ of his estate should belong to him, and the remainder to his mother; but if it were a daughter, he appointed the mother $\frac{2}{3}$, and the child the remainder. But it happened, that the addition was both a son and a daughter, by which the mother lost in equity 2400l. more than if it had been only a daughter. What would have been her dowry, had she had only a son? *Ans.* 2100l.

A young hare starts 40 yards before a grey-hound, and is not perceived by him till she has been up 40 seconds; she scuds away at the rate of ten miles an hour, and the dog, on view, makes after her at the rate of 18. How long will the course continue, and what will be the length of it from the place, where the dog set out? *Ans.* $60\frac{5}{22}$ seconds, and 530 yards run.

A reservoir for water has two cocks to supply it; by the first alone it may be filled in 40 minutes, by the second in 50 minutes, and it has a discharging cock, by which it may, when full, be emptied in 25 minutes. Now these three cocks being all left open, the influx and efflux of the water being always at the same rate, in what time would the cistern be filled?

Ans. 3 hours 20 minutes.

A sets out from London for Lincoln precisely at the time, when B at Lincoln sets out for London, distant 100 miles; after 7 hours they met on the road, and it then appeared, that A had ridden $1\frac{1}{2}$ mile an hour more than B. At what rate an hour did each of them travel? *Ans.* $7\frac{25}{28}$, B $6\frac{11}{28}$ miles.

What part of 3 pence is a third part of 2 pence. *Ans.* $\frac{2}{9}$.

A has by him $1\frac{1}{2}$cwt. of tea, the prime cost of which was 96l. sterling. Now interest being at 5 per cent. it is required to find how he must rate it per pound to B, so that by taking his negotiable note, payable at 3 months, he may clear 20 guineas by the bargain? *Ans.* 14s. $1\frac{1}{5}\frac{3}{9}$d. sterling.

There is an island 73 miles in circumference, and 3 footmen all start together to travel the same way about it; A goes 5 miles a day, B 8, and C 10; when will they all come together again? *Ans.* 73 days.

A man, being asked how many sheep he had in his drove, said, if he had as many more, half as many more, and 7 sheep and a half, he should have 20; how many had he? *Ans.* 5.

A person left 40s. to 4 poor widows, A, B, C, and D; to A he left $\frac{1}{3}$, to B $\frac{1}{4}$, to C $\frac{1}{5}$, and to D $\frac{1}{6}$, desiring the whole might be distributed accordingly; what is the proper share of each?

Ans. A's share 14s. $\frac{16}{38}$d. B's 10s. $6\frac{12}{38}$d. C's 8s. $5\frac{2}{38}$d. D's 7s. $\frac{8}{38}$d.

A general, disposing of his army into a square, finds he has

284 soldiers over and above; but increasing each side with one soldier, he wants 25 to fill up the square; how many soldiers had he? *Ans.* 24000.

There is a prize of 212l. 14s. 7d. to be divided among a captain, 4 men, and a boy; the captain is to have a share and a half; the men each a share, and the boy $\frac{1}{3}$ of a share; what ought each person to have?

Ans. The captain 54l. 14s. $\frac{3}{7}$d. each man 36l. 9s. 4$\frac{2}{7}$d. and the boy 12l. 3s. 1$\frac{3}{7}$d.

A cistern, containing 60 gallons of water, has 3 unequal cocks for discharging it; the greatest cock will empty it in one hour, the second in 2 hours, and the third in 3; in what time will it be emptied, if they all run together? *Ans.* 32$\frac{8}{11}$ minutes.

In an orchard of fruit trees, $\frac{1}{2}$ of them bear apples, $\frac{1}{4}$ pears, $\frac{1}{6}$ plums, and 50 of them cherries: how many trees are there in all? *Ans.* 600.

A can do a piece of work alone in ten days, and B in 13; if both be set about it together, in what time will it be finished?

Ans. 5$\frac{15}{23}$ days.

A, B, and C are to share 100000l. in the proportion of $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$, respectively; but C's part being lost by his death, it is required to divide the whole sum properly between the other two.

Ans. A's part is 57142$\frac{6}{7}$l. and B's 42857$\frac{1}{7}$l.

APPENDIX,

CONTAINING TABLES OF VARIOUS WEIGHTS AND MEASURES.

New French Weights and Measures.

THE weights and measures in common use are liable to great uncertainty and inconvenience. There being no fixed standard at hand, by which their accuracy can be ascertained, a great variety of measures, bearing the same name, has obtained in different countries. The foot, for instance, is used to stand for about a hundred different established lengths. The several denominations of weights and measures, are also arbitrary, and occasion most of the trouble and perplexity, that learners meet with in mercantile arithmetic.

To remedy these evils, the French government adopted a new system of weights and measures, the several denominations of which proceed in a decimal ratio, and all referable to a common permanent standard, established by nature, and accessible at all places on the earth. This standard is a meridian of the earth, which it was thought expedient to divide into 40 million parts. One of these parts is assumed as the unit of length, and the basis of the whole system. This they called a *metre*. It is equal to about $39\frac{1}{3}$ English inches, of which submultiples and multiples being taken, the various denominations of length are formed.

			Eng. Inch Dec.
A millimetre *is*	the 1000th part of a metre		,03937
A centimetre	the 100th part of a metre		,39371
A decimetre	the 10th part of a metre		3,93710
A METRE			39,37100
A decametre	10	metres	393,71000
A hecatometre	100	metres	3937,10000
A chiliometre	1000	metres	39371,00000
A myriometre	10000	metres	393710,00000
A grade or degree of the meridian equal to 100000 metres, or 100th part of the quadrant.			3937100,00000

	Mls.	Fur.	Yds.	Ft.	In. De.
The decametre is	0	0	10	2	9,7
The hecatometre	0	0	109	1	1
The chiliometre	0	4	213	1	10,2
The myriometre	6	1	156	0	6
The grade or decimal degree of the meridian	62	1	23	2	8

Measures of Capacity.

A cube, whose side is one tenth of a metre, that is, a cubic decimetre, constitutes the unit of measures of capacity. It is called the *litre*, and contains 61,028 cubic inches.

		Eng. Cub. In. Dec.
A millilitre or	1000th part of a litre	,06103
A centilitre	100th of a litre	,61028
A decilitre	10th of a litre	6,10280
A *litre*, a cubic	decimetre	61,02800
A decalitre	10 litres	610,28000
A hecatolitre	1000 litres	6102,80000
A chiliolitre	10000 litres	61028,00000
A myriolitre	100000 litres	610280,00000

The English pint, wine measure, contains 28,875 cubic inches. The litre therefore is 2 pints and nearly one eighth of a pint.

Hence

A decalitre is equal to 2 gal. $64\frac{44}{231}$ cubic inches.

A hecatolitre 26 gal. $4\frac{44}{231}$ cubic inches.

A chiliolitre 264 gal. $\frac{44}{231}$ cubic inches.

Weights.

The unit of weight is the *gramme*. It is the weight of a quantity of pure water, equal to a cubic centimetre, and is equal to 15,444 grains Troy.

	Gr. Dec.
A milligramme is 1000th part of a gramme	0,0154
A centigramme 100th of a gramme	0,1544
A decigramme 10th of a gramme	1,5444
A *gramme*, a cubic centimetre	15,4440
A decagramme 10 grammes	154,4402
A hecatogramme 100 grammes	1544,4023

A chilogramme	1000 grammes	15444,0234
A myriogramme	10000 grammes	154440,2344

A gramme being equal to 15,444 grains Troy.

A decagramme 6dwt. 10,44gr. equal to 5,65 drams Avoirdupois.

		lb.	oz.	dr.	
A hecatogramme	equal to	0	3	8,5	avoird.
A chilogramme		2	3	5	avoird.
A myriogramme		22	1	15	avoird.

100 myriogramms make a tun, wanting 32lb. 8oz.

Land Measure.

The unit is the *are,* which is a square decametre, equal to 3,95 perches. The deciare is a tenth of an are, the centiare is 100th of an are, and equal to a square metre. The milliare is 1000th of an are. The decare is equal to 10 ares; the hecatare to 100 ares, and equal to 2 acres 1 rood 35,4 perches English. The chilare is equal to 1000 ares, the myriare to 10000 ares.

For fire-wood the *stere* is the unit of measure. It is equal to a cubic metre, containing 35,3171 cubic feet English. The decestere is the tenth of a stere.

The quadrant of the circle generally is divided like the fourth part of the meridian, into 100 degrees, each degree into 100 minutes and each minute into 100 seconds, &c. so that the same number, which expresses a portion of the meridian, indicates also its length, which is a great convenience in navigation.

The coin also is comprehended in this system, and made to conform to their unit of weight. The weight of the *franc,* of which one tenth is alloy, is fixed at 5 grammes; its tenth part is called *décime,* its hundredth part *centime.*

The divisions of time, soon after the adoption of the above, underwent a similar alteration.

The year was made to consist of 12 months of 30 days each, and the excess of 5 days in common and 6 in leap years was considered as belonging to no month. Each month was divided into three parts, called decades. The day was divided into 10 hours, each hour into 100 minutes, and each minute into 100 seconds. This new calendar was adopted in 1793; in 1805 it was abolished, and the old calender restored. The weights and measures are still in use, and will probably prevail throughought the continent of Europe. They are recommended to the attention of every civilized country; and their general adoption would prove of no small importance to the scientific, as well as the commercial world.

Scripture Long Measure.

		Eng. Feet.	In. Dec.
4†	Digit	0	0,912
3	Palm	0	3,648
2	Span	0	10,944
4	Cubit	1	9,888
$1\frac{1}{2}$	Fathom	7	3,552
$1\frac{1}{3}$	Ezekiel's reed	10	11,328
10	Arabian pole	14	7,104
	Scoenus, measuring line	145	1,104

N. B. There was another span used in the East, equal to $\frac{1}{4}$th of a cubit.

Grecian Long Measure reduced to English.

		Eng. paces.	Feet.	In. Dec.
4	Dactylis, Digit	0	0	$0,7554\frac{11}{16}$
$2\frac{1}{2}$	Doron, Dochme, Palesta,	0	0	$3,0218\frac{3}{4}$
$1\frac{1}{10}$	Lichas	0	0	$7,5546\frac{7}{8}$
$1\frac{1}{11}$	Orthodoron	0	0	$8,3101\frac{9}{16}$
$1\frac{1}{3}$	Spithame	0	0	$9,0656\frac{1}{4}$
$1\frac{1}{8}$	Pous, foot	0	1	0,0875
$1\frac{1}{5}$	Pygme, cubit	0	1	$1,5984\frac{3}{8}$
$1\frac{1}{5}$	Pygon	0	1	$3,109\frac{3}{8}$
4	Pecus, cubit larger	0	1	6,13125
100	Orgya, pace	0	6	0,525
8	Stadium, Aulus ⊳ furlong	100	4	4,5
	Million, Mile	805	5	0

N. B. Two sorts of long measures were used in Greece, viz. the Olympic and the Pythic. The former was used in Peloponnesus, Attica, Sicily, and the Greek cities in Italy. The latter was used in Thessaly, Illyria, Phocis, and Thrace.

† These numbers show how many of each denomination it takes to make one of the next following.

The Olympic foot, properly called the Greek foot, according to
Dr. Hutton, contains 12,108 English inches,
Folkes, 12,072
Cavallo, 12,084

The Pythic foot, called also natural foot, according to
Hutton, contains 9,768
Paucton, 9,731

Hence it appears, that the Olympic stadium is 201½ English yards nearly; and the Pythic or Delphic stadium, 162½ yards nearly; and the other measures in proportion.

The Phyleterian foot is the Pythic cubit, or 1½ Pythic foot. The Macedonian foot was 13,92 English inches; and the Sicilian foot of Archimedes, 8,76 English inches.

Jewish Long or Itinerary Measure.

		Eng. Miles.	Paces.	Feet. Dec.
400	Cubit	0	0	1,824
5	Stadium	0	145	4,6
2	Sabbath day's journey	0	729	3,0
3	Eastern mile	1	403	1,0
8	Parasang	4	153	3,0
	A day's journey	33	172	4,0

Roman Long Measures reduced to English.

		Eng. Paces.	Feet.	In. Dec.
$1\frac{1}{3}$	Digitus traversus	0	0	$0,725\frac{3}{4}$
3	Uncia, or Inch	0	0	0,967
4	Palma minor	0	0	2,901
$1\frac{1}{4}$	Pes, or Foot	0	0	11,604
$1\frac{1}{5}$	Palmipes	0	1	2,505
$1\frac{2}{3}$	Cubitus	0	1	5,406
2	Gradus	0	2	5,01
125	Passus	0	4	10,02
8	Stadium	120	4	4,5
	Milliare	967	0	0

N. B. The Roman measures began with 6 scrupula = 1 sicilicum; 8 scrupula = 1 duellum; 1½ duellum = 1 seminaria; and 18 scrupula = 1 digitus. Two passus were equal to 1 decempeda.

Attic Dry Measures reduced to English.

		Pecks.	Gall.	Pints.	Sol. Inch.
10	Cochliarion	0	0	0	0,276$\frac{7}{26}$
1$\frac{1}{2}$	Cyathus	0	0	0	2,763$\frac{1}{2}$
4	Oxybaphon	0	0	0	4,144$\frac{3}{4}$
2	Cotylus	0	0	0	16,579
1$\frac{1}{2}$	Xestes, sextary	0	0	0	33,158
48	Chœnix	0	0	1	15,705$\frac{1}{4}$
	Medimnus	4	0	6	3,501

Attic Measures of Capacity for Liquids, reduced to English Wine Measure.

		Gal.	Pints.	Sol. In. Dec.
2	Cochliarion	0	$\frac{1}{120}$	0,0356$\frac{5}{12}$
1$\frac{1}{4}$	Cheme	0	$\frac{1}{60}$	0,0712$\frac{5}{6}$
2	Myston	0	$\frac{1}{48}$	0,089$\frac{11}{48}$
2	Concha	0	$\frac{1}{24}$	0,178$\frac{11}{24}$
1$\frac{1}{2}$	Cyathus	0	$\frac{1}{12}$	0,356$\frac{11}{21}$
4	Oxybathon	0	$\frac{1}{8}$	0,535$\frac{3}{8}$
2	Cotylus	0	$\frac{1}{2}$	2,141$\frac{1}{2}$
6	Xestes, sextary	0	1	4,283
12	Chous, congius	0	6	25,698
	Metretes, amphora	10	2	19;626

Others reckon 6 choi = 1 amphoreus, and 2 amphorei = 1 keramion or metretes. The keramion is stated by Paucton to have been equal to 35 French pints, or 8$\frac{2}{3}$ English gallons, and the other measures in proportion.

Measures of Capacity for Liquids, reduced to English Wine Measure.

		Gal.	Pints.	Sol. In. Dec.
4	Ligula	0	$\frac{1}{43}$	0,117$\frac{5}{12}$
1$\frac{1}{2}$	Cyathus	0	$\frac{1}{12}$	0,469$\frac{2}{3}$
2	Acetabulum	0	$\frac{1}{8}$	0,704$\frac{1}{2}$
2	Quartarius	0	$\frac{1}{4}$	1,409
2	Hemina	0	$\frac{1}{2}$	2,818
6	Sextarius	0	1	5,636
4	Congius	0	7	4,942
2	Urna	3	4$\frac{1}{2}$	5,33
20	Amphora	7	1	10,66
	Culeus	143	3	11,095

Jewish Dry Measures reduced to English.

		Pecks.	Gal.	Pints.	Sol. Inch.
20	Gachal	0	0	$01\frac{17}{20}$	0,031
$1\frac{4}{5}$	Cab	0	0	$2\frac{5}{6}$	0,073
$3\frac{1}{3}$	Gomor	0	0	$5\frac{1}{10}$	1,211
3	Seah	1	0	1	4,036
5	Epha	3	0	3	12,107
2	Letteeh	16	0	0	26,500
	Chomer, coron	32	0	1	18,969

Jewish Measures of Capacity for Liquids, reduced to English Wine Measure.

		Gal.	Pints.	Sol. Inch.
$1\frac{1}{3}$	Caph	0	$\frac{5}{8}$	0,177
4	Log	0	$\frac{5}{6}$	0,211
3	Cab	0	$3\frac{1}{3}$	0,844
2	Hin	1	2	2,533
3	Seah	2	4	5,067
10	Bath, epha	7	4	15,2
	Coron, chomer	75	5	7,625

Ancient Roman Land Measure.

100 Square Roman feet	= 1 Scrupulum of land
4 Scrupula	= 1 Sextulus
$1\frac{1}{5}$ Sextulus	= 1 Actus
6 Sextuli or 5 Actus	= 1 Uncia of land
6 Unciæ	= 1 Square Actus
2 Square Actus	= 1 Jugerum
2 Jugera	= 1 Heredium
100 Heredia	= 1 Centuria

N. B. If we take the Roman foot at 11,6 English inches, the Roman jugerum was 5980 English square yards, or 1 acre $37\frac{1}{2}$ perches.

Roman Dry Measures reduced to English.

		Peck.	Gal.	Pint.	Sol. In. De.
4	Ligula	0	0	$0\frac{1}{48}$	0,01
$1\frac{1}{2}$	Cyathus	0	0	$0\frac{1}{12}$	0,04
4	Acetabulum	0	0	$0\frac{1}{8}$	0,06
2	Hemina or Trutta	0	0	$0\frac{1}{2}$	0,24
8	Sextarius	0	0	1	0,48
2	Semi d.	0	1	0	3,84
	Modius	1	0	0	7,68

Table of the principal Gold and Silver Coins now current, containing their Weight, Fineness, Pure Contents, Current Value, and Intrinsic Value in Sterling, according to the Mint Price of England.

Country	Names of the Coins.	Weight.	Fineness car.	Fineness grs.	Pure Contents.	Current Value.	Value in Sterling. l.	s.	d.	Dolls.
	Gold Coins.	grs.	car.	grs.	grs.		l.	s.	d.	
Austrian Dominions,	Souverain, single	85,50	22		78,37	6 florins 40 creutzers		13	10	3,074
	Ducat Kremnitz *or* Hungarian	53,85	23	3	53,29	4 florins 30 creutzers		9	$5\frac{1}{4}$	2,097
Bavaria,	Carolin d'or	150,32	18	$2\frac{5}{6}$	117,18	10 florins 42 creutzers	1	0	9	4,611
	Max d'or	100,21	18	$2\frac{5}{6}$	78,12	7 florins 8 creutzers		13	10	3,076
Brunswick,	Carl d'or	102,36	21	3	92,76	5 rix dollars		16	$5\frac{1}{4}$	3,653
Bern,	Ducat	54,31	23	2	53,18	7 livres 4 sous		9	5	2,093
Denmark,	Ducat current	48.21	21	$0\frac{1}{3}$	42,35	12 marks Danish		7	6	1,667
East Indies,	Mohur, *or* gold rupee	176,50	23		169,15	15 silver rupees	1	9	$11\frac{1}{4}$	6,653
	Star pagoda	52,75	19	2	42,86	3 $\frac{3}{4}$ silver rupees		7	7	1,685
England,	Guinea	129,44	22		118,65	21 shillings	1	1	0	4,667
	Half guinea	64,72	22		59,32	10 $\frac{1}{2}$ shillings		10	6	2,333
	Seven shilling piece	43,15	22		39,55	7 shillings		7	0	1,556
Flanders,	See Austrian Dominions									
France,	Louis d'or, old, (coined before 1786)	125,51	21	$2\frac{1}{2}$	113,09	24 livres		19	$11\frac{3}{4}$	4,44
	Louis d'or, new, (coined since, 1786)	117,66	21	$2\frac{1}{2}$	106,02	24 livres		18	$9\frac{1}{4}$	4,171
	Napoleon, or piece of 40 francs, (new coins)	199,25	21	$0\frac{4}{5}$	179,33	40 francs	1	11	$8\frac{3}{4}$	7,051
Geneva,	Pistole	87,08	22		79,82	10 livres		14	$1\frac{1}{2}$	3,139
Genoa,	Sequin	53,90	23	$3\frac{1}{2}$	53,62	13 lire 10 soldi		9	$5\frac{3}{4}$	2,106
	Genovina d'oro	434,20	21	$3\frac{1}{2}$	396,74	100 lire	3	10	$2\frac{1}{2}$	15,597
	New piece of 96 lire	390,	21	$3\frac{1}{4}$	354,45	96 lire	3	2	9	13,945
Germany,	Ducat *ad legem Imperii*	53,85	23	$2\frac{1}{3}$	53,10	varies in different places		9	$4\frac{3}{4}$	2,088
Hamburgh,	See Germany									
Hanover	George d'or	103,03	21	3	93.37	5 rix dollars		16	$6\frac{1}{4}$	3,671
	Gold gulden	50,11	18		37,58	2 rix dollars		6	8	1,481